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THE TENSILE TUNNEL-CRACK WITH A SLIGHTLY WAVY FRONT

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Abstract—Several problems of three-dimensional fracture mechanics for a planar tunnel-crack loaded in pure mode I in an infinite elastic solid are investigated. The first is that of the bifurcation from the fundamental straight configuration of the crack front in brittle fracture, i.e. the possibility of the appearance of another, curved configuration still possessing the property that the stress intensity factor be constant along the front. The second is the stability of the same fundamental configuration versus small deviations from straightness within the crack plane in fatigue. The third is the (analytic at the start, but numerical *in fine*) determination of the fundamental "kernel" appearing in the integral expression of the variation of the stress intensity factor induced by a small perturbation of the crack front; this topic is considered after (and not before as would seem more natural) the first two in order to illustrate the fact that investigating the latter does not require a precise knowledge of that kernel as a necessary prerequisite. The last question envisaged is (again analytical first, but finally numerical) the calculation of the crack-face weight function in mode I for the crack configuration envisaged. Gao and Rice's previous works (1985, *ASME J. Appl. Mech.* **52,** 571–579; 1986, *ASME J. Appl. Mech.* **53,** 774–778; 1987, *Int. J. Fracture* **33,** 155–174; 1987, *ASME J. Appl. Mech.* **54,** 627–634; 1988, *Int. J. Solids Structures* **24,** 177–193) devoted to other crack shapes have been an important source of inspiration for this study with regard to both the topics investigated and some of the methods used.

1. INTRODUCTION

Consider (Fig. 1) a plane crack with arbitrary (geometrically regular) contour $\mathscr{F}$, located in an elastic body Ω and loaded in pure mode I through some symmetric system of tractions

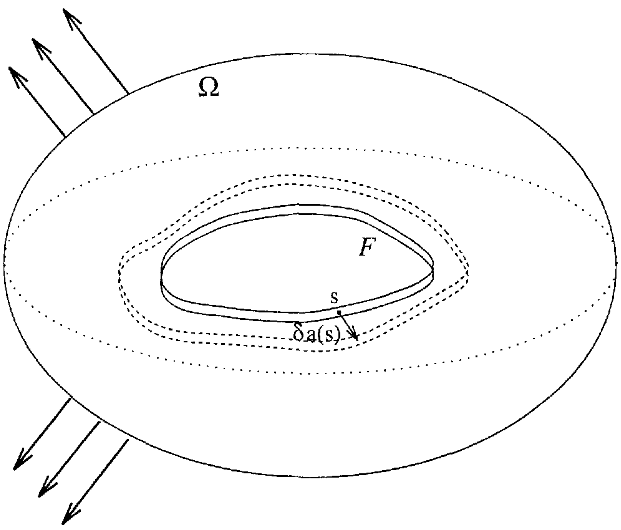

Fig. 1. Arbitrary plane tensile crack in an infinite body.

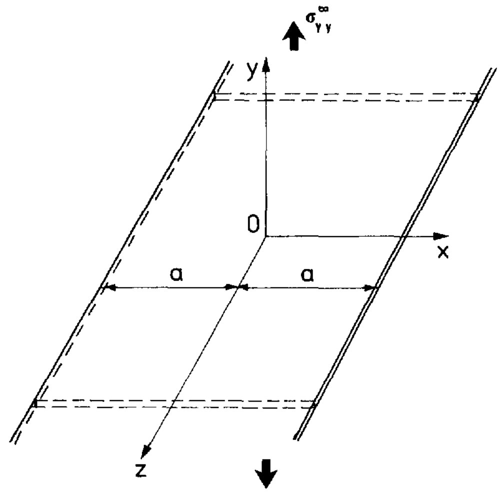

Fig. 2. Tensile tunnel-crack in an infinite body.

or displacements imposed on the outer boundary. The loading being kept constant, let the crack front be shifted within the crack plane, perpendicularly to itself, by a small distance $\delta a(s)$ depending (in a regular way) upon the position s ($\equiv$ curvilinear length along the front). Then the initial (mode I) stress intensity factor $k(s)$ at the point s changes by the amount $\delta k(s)$ given, to the first order in the perturbation, by the following formula:

$$\delta k(s) = [\delta k(s)]_{\delta a(s') \equiv \delta a(s)} + PV \int_{\mathscr{F}} Z(\Omega; s, s') k(s') [\delta a(s') - \delta a(s)]\, \mathrm{d}s', \tag{1}$$

where $[\delta k(s)]_{\delta a(s') \equiv \delta a(s)}$ denotes the value of $\delta k(s)$ for a uniform perturbation equal to $\delta a(s)$ and $Z(\Omega; s, s')$ is a function of s and s' which also depends upon the entire geometry of the body and the crack (including on which portions of the boundary forces versus displacements are prescribed), as symbolically indicated by the first argument. Some general properties of this function are as follows:

$$Z(\Omega; s, s') \sim \frac{1}{2\pi [D(s, s')]^2} \quad \text{for} \quad s' \to s; \quad Z(\Omega; s, s') = Z(\Omega; s', s) \quad \text{for all } s, s' \tag{2}$$

where $D(s, s')$ denotes the Cartesian distance between the points s and s'. The first property indicates that the integral in eqn (1) does have a meaning as a Cauchy principal value (PV).

The fundamental formula (1) was established by Zakharevitch (1985), Nazarov (1989) and Rice (1989). In the latter work, Rice in fact extended to arbitrary planar crack shapes some previous results obtained with Gao (Rice, 1985; Gao and Rice, 1986, 1987a,b; Gao, 1988) for various special cases: semi-infinite and internal circular cracks (loaded in mode I+II+III) and external circular crack (loaded in pure mode I). The work of Rice (1989) was itself very recently extended to cracks of completely arbitrary, non-planar shapes including possible kink angles (and also to arbitrary combinations of modes) by Mouchrif (1994). Property (2_1) was established in the work of Nazarov (1989) and was also apparent in the results of Gao and Rice, although Rice (1989) did not state $1/(2\pi)$ to be the universal limit of $Z(\Omega; s, s')[D(s, s')]^2$ for $s' \to s$, whatever the shape of the crack front. Property (2_2) was proved (for arbitrary plane cracks loaded in pure mode I) by both Nazarov (1989) and Rice (1989), and extended to arbitrary curved crack geometries and mixed modes by Mouchrif (1994).

The aim of the present work is to apply the preceding result to the study of various problems for a tunnel-crack in an infinite body loaded by some uniform opening stress σ_{yy}^{∞} at infinity (Fig. 2). The first one, which will be considered in Section 3, after some preliminaries, concerns the possible bifurcation from the straight configuration of the front in brittle fracture: can one find any configuration satisfying the condition that the stress intensity factor be equal to a constant (the fracture toughness) along the crack front, other than the "fundamental" one where both its front and rear parts remain straight (and move at opposite velocities)? The answer will turn out to be *yes*: there is a bifurcation mode (difference between two possible solutions) which corresponds to a sinusoidal shape of both parts of the front the "critical" wavelength of which is a characteristic multiple of the mean half-width a of the crack. This mode is symmetric with respect to the median axis Oz of the crack; there is no antisymmetric mode, except for a trivial global translatory motion of the two parts of the crack front in the direction x of propagation.

The second problem (which in fact bears a close relationship to the first one) will be studied in Section 4; it is that of the stability of the straight configuration of the crack front: that front being slightly but otherwise arbitrarily perturbed within the crack plane, will the perturbation subsequently decay or grow? This question is most readily addressed in the context of fatigue, assuming for instance the propagation law to be that of Paris. Performing a Fourier transform of the perturbation in the z-direction, one finds that to the first order in the amplitude of the perturbation, the evolutions of the various Fourier components are independent of each other; for a given wavelength, the difference of the Fourier components on the front and rear parts of the crack front always decays, but the sum grows or decays according to whether the wavelength is greater or smaller than the critical wavelength of the bifurcation mode evidenced previously. The latter observation is quite analogous to those made by Gao and Rice, and also Nguyen (1994), about similar problems. However, since the critical wavelength is proportional to the mean half-width of the crack and thus increases with time, the wavelength of any Fourier component finally becomes smaller than it and thus all Fourier components ultimately decay. Another noticeable point is that the curves representing (for a given wavelength) the sum and the difference of the Fourier components of the front and rear perturbations as functions of the mean half-width, which *a priori* depend upon two parameters, are in fact all equivalent modulo changes of origin and scale.

One remarkable feature of both the bifurcation and instability studies is that a major part of the analysis can be carried out without explicitly knowing the function $Z(\Omega;s,s')$ relevant to the case considered, the only ingredients required being a few simple and reasonable hypotheses about that function. It is, however, necessary to finally determine it notably in order to obtain the precise expression of the critical wavelength of bifurcation in brittle fracture. This is done in Section 5 in a somewhat more elegant way than in Mouchrif's (1994) thesis, by using an equation established by Rice (1989) which provides the variation of the function $Z(\Omega;s,s')$ induced by some slight variation of the domain occupied by the crack. This equation is applied here to special movements preserving the shape of the crack while changing its dimension and orientation. This procedure yields a second-order differential equation on the Fourier transform of the function $Z(\Omega;s,s')$; this equation is solved numerically and $Z(\Omega;s,s')$ is finally obtained through numerical Fourier inversion.

The full knowledge of the function $Z(\Omega;s,s')$ is finally used in Section 6 to compute the crack-face weight function of the tensile tunnel-crack. The principle of the method is to apply eqn (1) to the particular loading defining the weight function, considering the same special movements of the crack front as before. This procedure provides partial-differential equations on the Fourier transform of the weight function in the direction parallel to the crack front, which can be combined to yield an ordinary differential equation on each straight line parallel to that front. Again, this equation is solved numerically (on each such line) prior to final numerical Fourier inversion.

The methods employed in the last two sections are interesting in that they are of "special" rather than "general" nature in the terminology employed by Bueckner (1987). This means that they are "economical" in the sense that they avoid the calculation of the entire solution to the elasticity problems implied (namely that of a tunnel-crack with a slightly wavy front loaded by some uniform opening stress at infinity or with a straight front but a loading consisting of vertical point forces exerted on its lips), but concentrate on the sole feature of interest, namely the distribution of the stress intensity factor along the crack front. As a result, the treatment is simpler and more elegant, especially in the sense that the recourse to numerical computation involved is minimal and limited to some very final step. Also, it is quite clear that these methods can be employed to solve other problems of similar type; the calculation of the three-dimensional crack-face weight functions for a semi-infinite *interface* crack, for instance, will be envisaged in a future paper.

2. PRELIMINARIES

In the case considered here of a tunnel-crack, the unperturbed crack front consists of two straight lines; we shall therefore replace the curvilinear length s along it by the Cartesian coordinate $z^{\pm}$, where the upper index indicates whether the point of the front considered belongs to its front ($x = +a$) or rear ($x = -a$) part. Also, the only geometric parameter in the problem is the half-width a of the crack; it follows that the influence of the argument "Ω" upon the function $Z(\Omega; s, s')$ in fact reduces to a dependence of this function upon a. Furthermore, simple dimensional considerations show that $Z(\Omega; s, s') \equiv Z(a; z^{\pm}, z'^{\pm})$ is positively homogeneous of degree -2 with respect to its three arguments. Combining this feature with the obvious symmetries of the problem, one concludes that this function can be written in the following form:

$$\begin{cases} Z(a; z^{+}, z'^{+}) = Z(a; z^{-}, z'^{-}) \equiv \dfrac{f[(z'-z)/a]}{(z'-z)^2} \\ Z(a; z^{+}, z'^{-}) = Z(a; z^{-}, z'^{+}) \equiv \dfrac{g[(z'-z)/a]}{a^2} \end{cases} \tag{3}$$

where, in virtue of eqn (2), the functions f and g are bounded for $z' \to z$ and verify the following properties:

$$f(0) = \frac{1}{2\pi}; \quad f(-\eta) = f(\eta), \quad g(-\eta) = g(\eta) \quad \text{for all } \eta. \tag{4}$$

It follows that eqn (1) takes the form:

$$\delta k(z^{+}) = [\delta k(z^{+})]_{\delta a(z'^{\pm}) \equiv \delta a(z^{+})} + PV \int_{-\infty}^{+\infty} f\left(\frac{z'-z}{a}\right) k[\delta a(z'^{+}) - \delta a(z^{+})] \frac{\mathrm{d}z'}{(z'-z)^2}$$
$$+ \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right) k\delta a(z'^{-}) \frac{\mathrm{d}z'}{a^2} - \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right) k\delta a(z^{+}) \frac{\mathrm{d}z'}{a^2}$$

on the front part of the crack front, where $k \equiv \sigma_{yy}^{\infty}\sqrt{\pi a}$ is the initial uniform stress intensity factor; the split of the integral involving g into two parts here is licit since the integrand is non-singular for $z' \to z$. The variation of the stress intensity factor on the rear part of the front is given by the same formula but for the interchange of the upper indices $+$ and $-$. Now in the case where $\delta a(z^{\pm}) \equiv \delta a \equiv Cst$, the right-hand side reduces to its first term; since $(k + \delta k)(z^{+}) = \sigma_{yy}^{\infty}\sqrt{\pi(a + \delta a)}$ then, it follows that:

$$[\delta k(z^{+})]_{\delta a(z'^{\pm}) \equiv \delta a(z^{-})} = \sigma_{yy}^{\infty}\sqrt{\frac{\pi}{a}}\frac{\delta a}{2} = \frac{k\delta a}{2a}.$$

Also, if $\delta a(z^{+}) \equiv 0$ and $\delta a(z^{-}) \equiv \delta a \equiv Cst$, in which case $(k + \delta k)(z^{+}) = \sigma_{yy}^{\infty}\sqrt{\pi(a + \delta a/2)} \Rightarrow \delta k(z^{+}) = k\delta a/(4a)$, the expression of $\delta k(z^{+})$ reduces to its third term, so that

$$\int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right) k\delta a \frac{\mathrm{d}z'}{a^2} = \frac{k\delta a}{4a} \Rightarrow \int_{-\infty}^{+\infty} g(\eta)\,\mathrm{d}\eta = \tfrac{1}{4}. \tag{5}$$

Combining these elements, one puts the general formula for $\delta k(z^+)$ in the following form, which will serve as a basis for the analyses to follow:

$$\frac{\delta k(z^+)}{k} = \frac{\delta a(z^+)}{4a} + PV\int_{-\infty}^{+\infty} f\left(\frac{z'-z}{a}\right)[\delta a(z'^+) - \delta a(z^+)]\frac{\mathrm{d}z'}{(z'-z)^2}$$
$$+ \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right)\delta a(z'^-)\frac{\mathrm{d}z'}{a^2}. \tag{6}$$

In particular, for a perturbation of the form:

$$\delta a(z^+) \equiv \alpha\,\mathrm{e}^{\mathrm{i}pz}; \quad \delta a(z^-) \equiv \beta\,\mathrm{e}^{\mathrm{i}pz} \tag{7}$$

where α and β are complex amplitudes and p a real wavevector,† the expression of $\delta k(z^+)$ becomes, upon use of the change of variable $z' \equiv z + a\eta$:

$$\frac{\delta k(z^+)}{k} = [\alpha\bar{f}(q) + \beta\bar{g}(q)]\frac{\mathrm{e}^{\mathrm{i}pz}}{a}; \tag{8}$$

the "reduced" wavevector q and the functions $\bar{f}(q)$, $\bar{g}(q)$ in this equation are defined by:

$$q \equiv pa; \quad \bar{f}(q) \equiv \tfrac{1}{4} + PV\int_{-\infty}^{+\infty} f(\eta)(\mathrm{e}^{\mathrm{i}q\eta} - 1)\frac{\mathrm{d}\eta}{\eta^2} = \tfrac{1}{4} + 2\int_0^{+\infty} f(\eta)[\cos(q\eta) - 1]\frac{\mathrm{d}\eta}{\eta^2};$$
$$\bar{g}(q) \equiv \int_{-\infty}^{+\infty} g(\eta)\,\mathrm{e}^{\mathrm{i}q\eta}\,\mathrm{d}\eta = 2\int_0^{+\infty} g(\eta)\cos(q\eta)\,\mathrm{d}\eta \tag{9}$$

where use has been made of eqns (4_2) and (4_3). It should be noted that $\bar{f}(q)$ and $\bar{g}(q)$, just like $f(\eta)$ and $g(\eta)$, are even functions. Also, note that if the Fourier transform $\hat{\varphi}(q)$ of any function $\varphi(\eta)$ is defined by:

$$\hat{\varphi}(q) \equiv \int_{-\infty}^{+\infty} \varphi(\eta)\,\mathrm{e}^{\mathrm{i}q\eta}\,\mathrm{d}\eta \tag{10}$$

(without any factor $1/(2\pi)$ or $1/\sqrt{2\pi}$), $\bar{g}(q)$ is nothing else than the Fourier transform of $g(\eta)$ [$\bar{g}(q) \equiv \hat{g}(q)$]. The relation between $\bar{f}(q)$ and $\hat{f}(q)$ is a bit more complex. To derive it, it suffices to differentiate eqn (9_2) twice with respect to q; one then gets:

$$\bar{f}''(q) = -2\int_0^{+\infty} f(\eta)\cos(q\eta)\,\mathrm{d}\eta = -\int_{-\infty}^{+\infty} f(\eta)\,\mathrm{e}^{\mathrm{i}q\eta}\,\mathrm{d}\eta \equiv -\hat{f}(q). \tag{11}$$

Let us finally stress that although we shall not state the explicit values of the functions $f(\eta)$, $g(\eta)$, $\bar{f}(q)$, $\bar{g}(q)$ appearing in the fundamental formulae (6) and (8) before Section 5, they are completely determined by the geometry considered and must thus be regarded as data, not unknowns.

† The calculation of $\delta k(z^+)$ for a complex perturbation here is purely formal, and will serve in the sequel only to derive the value of $\delta k(z^+)$ for actual, real perturbations through linear superposition; the result will of course be real then.

3. BIFURCATION OF THE CRACK FRONT FROM ITS FUNDAMENTAL STRAIGHT CONFIGURATION IN BRITTLE FRACTURE

Investigating the bifurcation problem requires the expression of the time-derivative of the stress intensity factor on the front and rear parts of the crack front at some instant when they are straight but their velocities are non-uniform. Formula (6) does provide the desired expression of $\dot{k}(z^+)$ [it suffices to replace $\delta k(z^+)$, $\delta a(z^+)$ and $\delta a(z'^{\pm})$ by $\dot{k}(z^+)$, $\dot{a}(z^+)$ and $\dot{a}(z'^{\pm})$], but only for a constant loading (as mentioned in Section 1), whereas quasi-static propagation of the crack in brittle fracture requires it to vary with time.† To extend this expression to the general case of some variable loading, it suffices to add to that part of $\dot{k}(z^+)$ due to the motion of the crack front, that arises from the variation of the loading, i.e. $\dot{\sigma}_{yy}^{\infty}\sqrt{\pi a} = k\dot{\sigma}_{yy}^{\infty}/\sigma_{yy}^{\infty}$; this yields:

$$\frac{\dot{k}(z^+)}{k} = \frac{\dot{a}(z^+)}{4a} + PV\int_{-\infty}^{+\infty} f\left(\frac{z'-z}{a}\right)[\dot{a}(z'^+)-\dot{a}(z^+)]\frac{dz'}{(z'-z)^2} + \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right)\dot{a}(z'^-)\frac{dz'}{a^2} + \frac{\dot{\sigma}_{yy}^{\infty}}{\sigma_{yy}^{\infty}}.$$

In order for the stress intensity factor to remain equal to k_c at all locations just after the instant considered, the right-hand side here must be zero. The problem is to see whether the solution to the resulting integral equation on the functions $\dot{a}(z^{\pm})$, supplemented by its analog on the rear part of the crack front, is unique or not. The difference of two possible solutions, which we shall call a *bifurcation mode* and still denote $\dot{a}(z^{\pm})$ for simplicity, verifies the corresponding homogeneous equation:

$$\frac{\dot{a}(z^+)}{4a} + PV\int_{-\infty}^{+\infty} f\left(\frac{z'-z}{a}\right)[\dot{a}(z'^+)-\dot{a}(z^+)]\frac{dz'}{(z'-z)^2} + \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right)\dot{a}(z'^-)\frac{dz'}{a^2} = 0 \qquad (12)$$

(plus its analog on the rear part of the front). The problem here is to see whether non-zero solutions can be found.

Because of the invariance of the problem in the z-direction, it is natural to introduce the Fourier transforms of a possible bifurcation mode on the two parts of the crack front, in the form:‡

$$\dot{a}(z^+) \equiv \int_{-\infty}^{+\infty} \alpha(p)e^{ipz}\,dp;\quad \dot{a}(z^-) \equiv \int_{-\infty}^{+\infty} \beta(p)e^{ipz}\,dp. \qquad (13)$$

The integral equation (12) then becomes, by eqns (7) and (8):

$$\int_{-\infty}^{+\infty} [\alpha(p)\bar{f}(q)+\beta(p)\bar{g}(q)]\frac{e^{ipz}}{a}dp = 0 \quad (\forall z),$$

where $q \equiv pa$ as above. This equation and its analog on the rear part of the front are equivalent to:

† Consider for instance the case where both parts of the crack front remain straight; then, at every instant, $k(z^{\pm}) = \sigma_{yy}^{\infty}\sqrt{\pi a} \equiv k_c \equiv Cst$, so that σ_{yy}^{∞} must decrease if a is to increase.

‡ With the notation of eqn (10), one has $\alpha(p) \equiv (1/(2\pi))\widehat{\dot{a}(z^+)}(-p)$ and $\beta(p) \equiv (1/(2\pi))\widehat{\dot{a}(z^-)}(-p)$; but the notations $\alpha(p)$ and $\beta(p)$ will be preferred for simplicity.

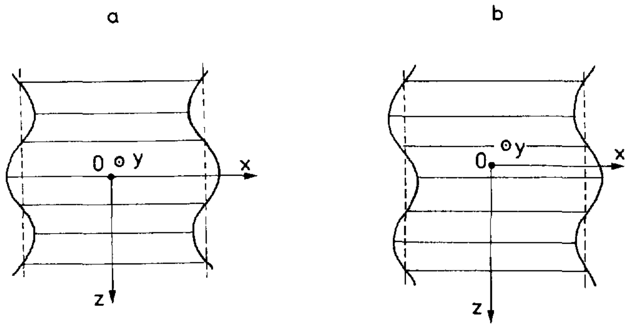

Fig. 3. Bifurcation modes in brittle fracture: (a) symmetric mode; (b) antisymmetric mode.

$$\begin{cases} \bar{f}(q)\alpha(p)+\bar{g}(q)\beta(p) = 0 \\ \bar{g}(q)\alpha(p)+\bar{f}(q)\beta(p) = 0 \end{cases} (\forall p). \tag{14}$$

We are looking for non-zero "Fourier components" $[\alpha(p), \beta(p)]$ of a possible bifurcation mode. For such components, the determinant $\bar{f}^2(q)-\bar{g}^2(q)$ of the above system must be zero. Hence the reduced wavelength q must be equal to $\pm q_s$ or $\pm q_a$, where q_s and q_a are the positive solutions (dimensionless numbers) of the equations:

$$\bar{f}(q_s)+\bar{g}(q_s) = 0; \quad \bar{f}(q_a)-\bar{g}(q_a) = 0 \tag{15}$$

(the existence and uniqueness of these solutions will be established shortly). If $q = \pm q_s \Rightarrow p = \pm q_s/a$, system (14) reduces to $\alpha(p)-\beta(p) = 0 \Rightarrow \alpha(p) = \beta(p)$: the bifurcation mode is symmetric (see Fig. 3a). If $q = \pm q_a \Rightarrow p = \pm q_a/a$, eqn (14) reduces to $\alpha(p) = -\beta(p)$ and the mode is antisymmetric (Fig. 3b).†

To pursue the analysis, it becomes necessary to introduce a few hypotheses concerning the functions $f(\eta), g(\eta), \bar{f}(q)$ and $\bar{g}(q)$; these hypotheses are reasonable and will be confirmed later when these functions are evaluated numerically. First, let us note that making the change of variable $\eta' = q\eta$ in the definition (9_2) of $\bar{f}(q)$ and assuming $f(\eta)$ to be bounded, one sees that the integral term in that definition vanishes for $q \to 0$ so that $\bar{f}(0) = 1/4$; combining the same change of variable and (4_1), one also concludes that $\lim_{q\to+\infty} \bar{f}(q) = -\infty$. Furthermore it follows from eqns (5) and (9_3) that the value of $\bar{g}(0)$ is also 1/4, and the latter equation also implies that $\lim_{q\to+\infty} \bar{g}(q) = 0$, the function $g(\eta)$ being supposed to be regular. If one makes the additional assumption that the functions $\bar{f}(q)$, $\bar{g}(q)$ and $\bar{f}(q)-\bar{g}(q)$ are monotone on the interval $[0, +\infty]$, one may conclude that both $\bar{f}(q)$ and $\bar{g}(q)$ decrease on $[0, +\infty]$, from 1/4 to $-\infty$ and 1/4 to 0, respectively, $\bar{f}(q)$ remaining always smaller than $\bar{g}(q)$. These features are schematically represented in Fig. 4. It immediately follows that eqns (15) do have unique solutions, and that:

$$0 = q_a < q_s. \tag{16}$$

The first property here means that the wavelength $\lambda_a \equiv 2\pi a/q_a$ of the antisymmetric mode is infinite, so that the latter in fact represents a mere translatory motion of the two parts of the front in the x-direction. The existence of such a bifurcation mode is a trivial feature arising from the fact that for a straight crack front, the stress intensity factor depends only on the width of the crack, i.e. on the relative positions of the front and rear parts of the front in the x-direction and not on their absolute positions. The symmetric mode, on the other hand, is non-trivial; the explicit calculation of its wavelength $\lambda_s \equiv 2\pi a/q_s$ will be carried out in Section 5.

† Figure 3 may seem paradoxical at first sight because of the backward motion of some points of the crack front; to resolve the "paradox", it suffices to remember that a bifurcation mode does not represent an actual movement of the crack front, but the difference between two such movements.

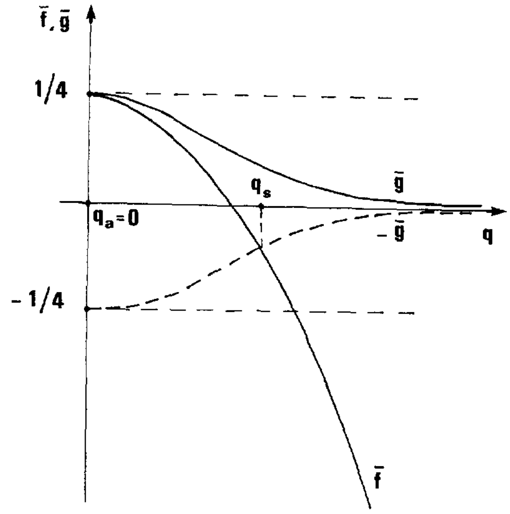

Fig. 4. Qualitative shape of the functions $\bar{f}(q)$ and $\bar{g}(q)$.

4. STABILITY OF THE STRAIGHT CONFIGURATION OF THE CRACK FRONT IN FATIGUE

Here the crack will be assumed to propagate in fatigue, the propagation rate being given by Paris' simple law:

$$\frac{\partial a}{\partial N} = C(\Delta k)^n \tag{17}$$

where N denotes the number of cycles, Δk the amplitude of variation of the stress intensity factor during one cycle, and C and n positive material constants. The amplitude of the loading will not necessarily be constant in time. We shall assume the crack front to slightly depart, at every instant t, from a strictly straight configuration corresponding to some half-width $a(t)$. We shall use the Fourier transforms of the perturbations on the two parts of the crack front: the distances $(a+\delta a)(z^{\pm}, t)$ from the median axis Oz to the front and rear parts of the front will be written in the form:†

$$\begin{aligned} (a+\delta a)(z^+, t) &\equiv a(t)+\int_{-\infty}^{+\infty} \alpha(p,t)\,\mathrm{e}^{ipz}\,\mathrm{d}p; \\ (a+\delta a)(z^-, t) &\equiv a(t)+\int_{-\infty}^{+\infty} \beta(p,t)\,\mathrm{e}^{ipz}\,\mathrm{d}p. \end{aligned} \tag{18}$$

For any time t, let $\Delta k(t)$ denote the (spatially uniform) amplitude of the stress intensity factor for the unperturbed tunnel-crack of half-width $a(t)$, and $\delta\Delta k(z^{\pm}, t)$ the correction arising from the perturbation; by eqns (7) and (8),

$$\Delta k(t)+\delta\Delta k(z^+, t) = \Delta k(t)\left(1+\int_{-\infty}^{+\infty}\{\alpha(p,t)\bar{f}[pa(t)]+\beta(p,t)\bar{g}[pa(t)]\}\frac{\mathrm{e}^{ipz}}{a(t)}\mathrm{d}p\right).$$

† Again, with the notation of eqn (10), $\alpha(p) \equiv (1/(2\pi))\widehat{\delta a(z^+)}(-p)$ and $\beta(p) \equiv (1/(2\pi))\widehat{\delta a(z^-)}(-p)$.

The value of the propagation rate of the front part of the crack front is therefore

$$C[\Delta k(N)+\delta\Delta k(z^+,N)]^n = C[\Delta k(N)]^n\left(1+n\int_{-\infty}^{+\infty}\{\bar{f}[pa(N)]\alpha(p,N) + \bar{g}[pa(N)]\beta(p,N)\}\frac{e^{ipz}}{a(N)}\,dp\right)$$

to the first order in the perturbation, where t has been replaced by the more relevant argument N. Equating this expression to:

$$\frac{\partial(a+\delta a)}{\partial N}(z^+,N) \equiv \frac{da}{dN}(N)+\int_{-\infty}^{+\infty}\frac{\partial\alpha}{\partial N}(p,N)\,e^{ipz}\,dp,$$

identifying terms independent of z and proportional to e^{ipz} and supplementing the resulting equations with their analogs on the rear part of the crack front, one obtains:

$$\begin{cases} \dfrac{da}{dN}(N) = C[\Delta k(N)]^n \\ \left.\begin{aligned} \dfrac{\partial\alpha}{\partial N}(p,N) &= \dfrac{nC}{a(N)}[\Delta k(N)]^n\{\bar{f}[pa(N)]\alpha(p,N)+\bar{g}[pa(N)]\beta(p,N)\} \\ \dfrac{\partial\beta}{\partial N}(p,N) &= \dfrac{nC}{a(N)}[\Delta k(N)]^n\{\bar{g}[pa(N)]\alpha(p,N)+\bar{f}[pa(N)]\beta(p,N)\} \end{aligned}\right\} \quad (\forall p). \end{cases}$$

In the right-hand sides of the evolution equations for $\alpha(p,N)$ and $\beta(p,N)$ here, the functions $\alpha(p',N)$ and $\beta(p',N)$ appear only through their values at the point $p'=p$. This means that although for a given wavevector p, the evolutions of the Fourier components $\alpha(p,N)$ and $\beta(p,N)$ are coupled, those of Fourier components corresponding to different wavevectors are independent.

Provided that one knows how the amplitude of the loading varies with N, Δk is known as a function of a and N; it is then theoretically possible (at least numerically) to integrate the above system and get a, α and β (for any value of p) as functions of N. However, determining α and β as functions of a (for a given p) is more interesting, because it turns out that their expressions are then not only remarkably simple but also independent of the temporal variation of the loading and the value of the Paris constant C. Indeed, eliminating dN in the above equations, one finds:

$$\begin{cases} \dfrac{\partial\alpha}{\partial a}(p,a) = \dfrac{n}{a}[\bar{f}(pa)\alpha(p,a)+\bar{g}(pa)\beta(p,a)] \\ \dfrac{\partial\beta}{\partial a}(p,a) = \dfrac{n}{a}[\bar{g}(pa)\alpha(p,a)+\bar{f}(pa)\beta(p,a)]; \end{cases}$$

taking the sum and the difference of these equations and integrating, we get:

$$\begin{cases} \dfrac{\alpha(p,a)+\beta(p,a)}{\alpha_0(p)+\beta_0(p)} = \exp\left\{n\displaystyle\int_{pa_0}^{pa}[\bar{f}(q)+\bar{g}(q)]\frac{dq}{q}\right\} \\ \dfrac{\alpha(p,a)-\beta(p,a)}{\alpha_0(p)-\beta_0(p)} = \exp\left\{n\displaystyle\int_{pa_0}^{pa}[\bar{f}(q)-\bar{g}(q)]\frac{dq}{q}\right\} \end{cases} \tag{19}$$

where the subscript $_0$ indicates initial values.

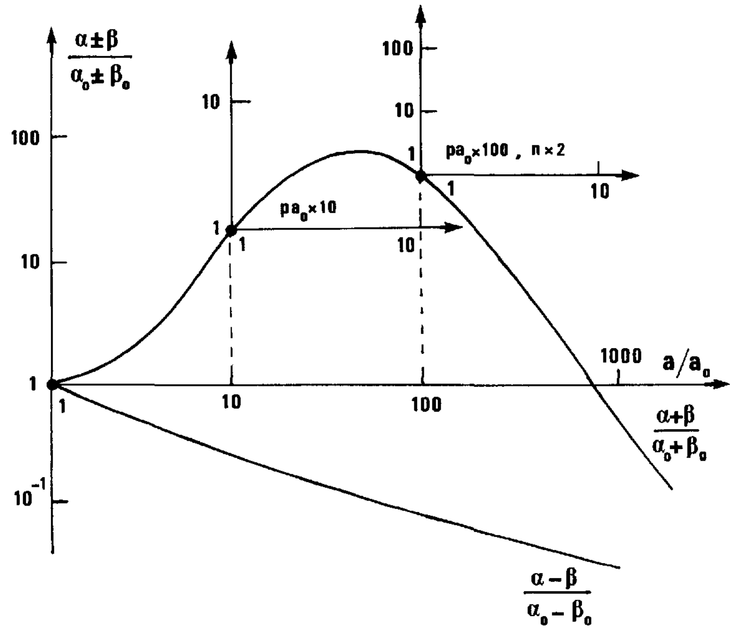

Fig. 5. Qualitative shape of the curves $(\alpha\pm\beta)/(\alpha_0\pm\beta_0) = f(a/a_0)$ in fatigue and changes of origin and scale on these curves.

These formulae allow for an easy discussion of stability. Let us assume the wavevector p to be positive (the discussion for $p < 0$ is similar, since the right-hand sides in eqns (19) are even functions of p). We have seen that $\bar{f}(q)-\bar{g}(q)$ is negative; it follows that the integral in eqn (19_2) is a decreasing function of a and therefore that the ratio $(\alpha-\beta)/(\alpha_0-\beta_0)$ decreases in time; in other words, *whatever the value of the wavevector considered, if the corresponding initial Fourier components of the perturbation on the front and rear parts of the crack front are different, they will necessarily subsequently tend to become equal.*

Furthermore, Fig. 4 makes it clear that $\bar{f}(q)+\bar{g}(q)$ is positive for $q < q_s$ and negative for $q > q_s$. Thus the integral in eqn (19_1) and the ratio $(\alpha+\beta)/(\alpha_0+\beta_0)$ are increasing functions of a for $pa < q_s$ and decreasing functions for $pa > q_s$. This means that *stability or instability prevails for the sum of the Fourier components of the perturbation on the front and rear parts of the crack front according to whether the wavelength $\lambda \equiv 2\pi/p$ of these components is smaller or greater than the "critical" wavelength $\lambda_s \equiv 2\pi a/q_s$.* This instability phenomenon for large wavelengths was already anticipated in the work of Rice (1985), although the minimum value of the wavelength for its occurrence was in fact infinite in the case of a semi-infinite crack ($a = +\infty$) considered there. The same phenomenon was also observed by Gao and Rice (1987a,b) for internal and external circular cracks subjected to appropriate loadings, and by Nguyen (1994) for the unsticking of an infinite tight membrane stuck onto a rigid plate except on a strip of finite width and infinite length loaded by an internal pressure.

It must be remarked, however, that since the critical wavelength is proportional to the mean half-width of the crack and thus continuously increases, stability ultimately prevails for all wavelengths. There are in fact two cases. If the wavelength $\lambda \equiv 2\pi/p$ of the Fourier components considered is smaller than the initial critical wavelength $\lambda_{s0} \equiv 2\pi a_0/q_s$, then the ratio $(\alpha+\beta)/(\alpha_0+\beta_0)$ will always decrease. If, conversely, λ is larger than λ_{s0}, that ratio will increase until λ_s reaches the value λ, then subsequently decrease.

Figure 5, based on these considerations, shows the qualitative shape of the curves $(\alpha\pm\beta)/(\alpha_0\pm\beta_0) = f(a/a_0)$ (in the second case).† One remarkable property of these curves, represented in a log–log plot as is done here, is that although they depend upon two

† It must be emphasized that the numbers in this figure do not in any way mean that the representation of the curves is exact; they are there merely to illustrate the changes of origin and scale discussed below.

parameters, n and pa_0 (as can be seen by writing the upper bound in the integrals of eqns (19) in the form $(pa_0)(a/a_0)$), they are universal in the sense that it is sufficient to know them for some special choice of these parameters, to know them in all cases. Indeed it immediately results from eqns (19) that for a given curve $(\alpha+\beta)/(\alpha_0+\beta_0) = f(a/a_0)$ or $(\alpha-\beta)/(\alpha_0-\beta_0) = f(a/a_0)$:

1. If pa_0 is multiplied by some factor, one gets the new curve by simply shifting the origin (i.e. the point $(\alpha\pm\beta)/(\alpha_0\pm\beta_0) = 1, a/a_0 = 1$) horizontally by the same factor, and vertically in such a way that it remain on the curve.
2. If n is multiplied by some factor, the new curve is even more simply obtained by changing the vertical (logarithmic) scale by the same factor.

These transformations are schematically illustrated in Fig. 5.

It finally remains to derive conditions on the initial perturbation ensuring that $|\delta a(z^\pm, t)|$ will always remain much smaller than $a(t)$, as required for the first-order analysis to be valid. It is equivalent to request that $|\delta a(z^+, t)+\delta a(z^-, t)|$ and $|\delta a(z^+, t)-\delta a(z^-, t)|$ satisfy the same condition. Sufficient conditions for this to be true are:

$$\int_{-\infty}^{+\infty} |\alpha(p, t)+\beta(p, t)|\,\mathrm{d}p \ll a(t)\,; \int_{-\infty}^{+\infty} |\alpha(p, t)-\beta(p, t)|\,\mathrm{d}p \ll a(t). \tag{20}$$

Condition (20_2) is true for any t as soon as it holds at $t = 0$, since $a(t)$ continuously increases and for any p, $|\alpha(p, t)-\beta(p, t)|$ is a decreasing function of time.

The situation is more complex for the condition (20_1). If $\lambda < \lambda_{s0}$, $|\alpha(p, t)+\beta(p, t)|$ continuously decreases; thus:

$$\lambda < \lambda_{s0} \Rightarrow \frac{|\alpha(p, t)+\beta(p, t)|}{a(t)} \leqslant \frac{|\alpha_0(p)+\beta_0(p)|}{a_0}. \tag{21}$$

If $\lambda > \lambda_{s0}$, $|\alpha(p, t)+\beta(p, t)|$ increases in a first phase and decreases in a second one; it follows that the maximum of $|\alpha(p, t)+\beta(p, t)|/a(t)$ occurs during the first phase. In order to bound that maximum, let us note that since $\bar{f}(q)+\bar{g}(q) \leqslant 1/2$ (see Fig. 4), eqn (19_1) implies that:

$$\frac{|\alpha(p, a)+\beta(p, a)|}{|\alpha_0(p)+\beta_0(p)|} \leqslant \left(\frac{a}{a_0}\right)^{n/2} \Rightarrow \frac{|\alpha(p, t)+\beta(p, t)|}{a(t)} \leqslant \frac{|\alpha_0(p)+\beta_0(p)|}{a_0}\left(\frac{a(t)}{a_0}\right)^{n/2-1};$$

now the Paris exponent is always greater than 2 in practice, so that the maximum of $[a(t)/a_0]^{n/2-1}$ during the first phase is reached when $a(t)/a_0$ is itself maximum, i.e. at the end of that phase, and its value is then $[q_s/(pa_0)]^{n/2-1} \equiv (\lambda/\lambda_{s0})^{n/2-1}$. It follows that:

$$\lambda > \lambda_{s0} \Rightarrow \frac{|\alpha(p, t)+\beta(p, t)|}{a(t)} \leqslant \frac{|\alpha_0(p)+\beta_0(p)|}{a_0}\left(\frac{\lambda}{\lambda_{s0}}\right)^{n/2-1}. \tag{22}$$

Equations (21) and (22) can be summarized in the following single inequality, valid for all values of λ:

$$\frac{|\alpha(p, t)+\beta(p, t)|}{a(t)} \leqslant \frac{|\alpha_0(p)+\beta_0(p)|}{a_0}\cdot \mathrm{Max}\left[1, \left(\frac{\lambda}{\lambda_{s0}}\right)^{n/2-1}\right],$$

and it follows that the condition:

$$\int_{-\infty}^{+\infty} \frac{|\alpha_0(p)+\beta_0(p)|}{a_0} \cdot \mathrm{Max}\left[1, \left(\frac{\lambda(p)}{\lambda_{s0}}\right)^{n/2-1}\right] \cdot \mathrm{d}p \ll 1$$

[where $\lambda(p) \equiv 2\pi/|p|$] suffices to ensure that eqn (20_1) will be satisfied for all t.

In conclusion, the conditions looked for are:

$$\int_{-\infty}^{+\infty} |\alpha_0(p)-\beta_0(p)|\,\mathrm{d}p \ll a_0\,; \int_{-\infty}^{+\infty} |\alpha_0(p)+\beta_0(p)| \cdot \mathrm{Max}\left[1, \left(\frac{\lambda(p)}{\lambda_{s0}}\right)^{n/2-1}\right] \cdot \mathrm{d}p \ll a_0. \tag{23}$$

5. CALCULATION OF THE FUNCTIONS $\bar{f}(q)$, $\bar{g}(q)$, $f(\eta)$ AND $g(\eta)$

We shall now proceed to the actual calculation of the functions $\bar{f}(q)$ and $\bar{g}(q)$. This will allow us, not only to explicitly calculate the critical wavelength of bifurcation λ_s in brittle fracture and the curves $(\alpha+\beta)/(\alpha_0 \pm \beta_0) = f(a/a_0)$ in fatigue, but also to evaluate the functions $f(\eta)$ and $g(\eta)$ [i.e. the fundamental "kernel" $Z(\Omega\,; s, s') \equiv Z(a\,; z^{\pm}, z'^{\pm})$] and, as a bonus, the crack-face weight function in pure mode I for the crack geometry considered.

The method that will be used is based on an equation established by Rice (1989) which gives the first-order variation of the fundamental kernel $Z(\Omega\,; s, s')$ induced by some slight perturbation $\delta a(s)$ of the crack front within the crack plane. This equation reads:

$$\delta Z(\Omega\,; s_1, s_2) = PV \int_{\mathscr{F}} Z(\Omega\,; s_1, s) Z(\Omega\,; s, s_2) \delta a(s)\,\mathrm{d}s \tag{24}$$

where s_1 and s_2 are any *immobile* [$\delta a(s_1) = \delta a(s_2) = 0$] points of the crack front.

Rice's view was that eqn (24) could be employed to determine the kernel $Z(\Omega\,; s, s')$ for new crack shapes through numerical integration, starting from some reference shape for which it would be known. The use we make of it here is somewhat different: instead of studying motions of the crack front generating new crack shapes, we only consider motions preserving the original one, though possibly modifying the size and orientation of the crack. This procedure yields integro-differential equations on the fundamental kernel.

A comparison with the method employed in Mouchrif's (1994) thesis is given in Appendix A for the sake of completeness. Mouchrif's method is somewhat more heavy than that used here (as already mentioned in Section 1), although some of the information it provides is slightly more precise (see below). The final numerical results obtained are very similar.

5.1. *Integro-differential equations on the functions* f(η) *and* g(η)

Let us first consider (Fig. 6a) a simple translatory motion of the sole rear part of the crack front, defined by $\delta a(z^+) \equiv 0$, $\delta a(z^-) \equiv \varepsilon$ where ε denotes a small parameter; then, by eqn (3), eqn (24) reads for arbitrary points z_1^+, z_2^+ of the front part of the front:

$$\varepsilon \frac{\partial}{\partial \varepsilon}\left\{\frac{f[(z_1-z_2)/(a+\varepsilon/2)]}{(z_1-z_2)^2}\right\}_{\varepsilon=0} = \int_{-\infty}^{+\infty} g\left(\frac{z_1-z}{a}\right) g\left(\frac{z-z_2}{a}\right) \frac{\varepsilon\,\mathrm{d}z}{a^4},$$

or equivalently

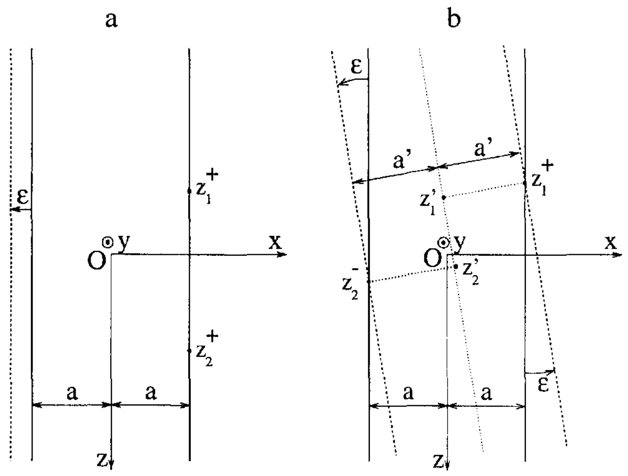

Fig. 6. Special motions of the crack front: (a) translatory movement of the rear part; (b) rotation of both parts by the same angle.

$$f'(\eta) = -2\eta \int_{-\infty}^{+\infty} g(\eta')g(\eta-\eta')\,\mathrm{d}\eta' \tag{25}$$

where the changes of variables $\eta \equiv (z_1 - z_2)/a$ and $\eta' \equiv (z_1 - z)/a$ have been used.

Now consider a perturbation of the crack front defined by a rotation of its front and rear parts by the same angle ε about the points z_1^+ and z_2^- (Fig. 6b). Then eqn (24) reads

$$\varepsilon\frac{\partial}{\partial\varepsilon}\left\{\frac{g[(z_1'-z_2')/a']}{a'^2}\right\}_{\varepsilon=0} = PV\int_{-\infty}^{+\infty}\frac{f[(z_1-z)/a]}{(z_1-z)^2}\frac{g[(z-z_2)/a]}{a^2}\varepsilon(z-z_1)\,\mathrm{d}z$$
$$+PV\int_{-\infty}^{+\infty}\frac{g[(z_1-z)/a]}{a^2}\frac{f[(z-z_2)/a]}{(z-z_2)^2}\varepsilon(z_2-z)\,\mathrm{d}z$$

where z_1' and z_2' denote the abscissae of the projections of the points z_1^+, z_2^- onto an axis parallel to the new crack front, and a' the new half-width of the crack. It is easy to show that

$$a' = a - \frac{z_1-z_2}{2}\varepsilon + O(\varepsilon^2)\,;\, z_1'-z_2' = z_1-z_2+2a\varepsilon+O(\varepsilon^2),$$

and it follows upon use of the same changes of variables as above that the preceding equation reads

$$\left[\left(1+\frac{\eta^2}{4}\right)g(\eta)\right]' = -PV\int_{-\infty}^{+\infty}\frac{f(\eta')}{\eta'}g(\eta-\eta')\,\mathrm{d}\eta'. \tag{26}$$

5.2. *Differential equation on the function* $\bar{g}(q)$

The presence of the convolution products in eqns (25) and (26) strongly suggests the use of their Fourier transforms. This immediately yields for eqn (25):

$$q\hat{f}(q) = -4\bar{g}(q)\bar{g}'(q), \tag{27}$$

the Fourier transform being defined by eqn (10) [it is reminded that with the notation of that equation, $\bar{g}(q) \equiv \hat{g}(q)$].

Fourier-transforming eqn (26) is slightly more difficult. Let us put $f(\eta) \equiv \eta h(\eta)$. Then $\hat{f}(q) = \widehat{\eta h}(q) = -i\hat{h}'(q)$, the integral defining the Fourier transform of $h(\eta)$ being understood in the sense of a Cauchy principal value. It follows that $\widehat{f/\eta}(q) = \hat{h}(q) = i\hat{F}(q)$ where $\hat{F}(q)$ is an indefinite integral of $\hat{f}(q)$; to determine the arbitrary constant involved in its definition, it suffices to note that $\hat{h}(0) = 0$ since $f(\eta)$ is an even function; thus $\hat{F}(0) = 0$, which means that $\hat{F}(q)$ is the unique *odd* indefinite integral of $\hat{f}(q)$. One then has $(\widehat{f/\eta * g})(q) = \widehat{f/\eta}(q)\bar{g}(q) = i\hat{F}(q)\bar{g}(q)$ where $*$ denotes the convolution product and the integral defining $f/\eta * g$ is to be understood in the sense of a principal value, so that the Fourier transform of eqn (26) reads

$$q\left[\bar{g}(q) - \frac{\bar{g}''(q)}{4}\right] = \hat{F}(q)\bar{g}(q). \tag{28}$$

Quite remarkably, it is possible to explicitly integrate the system (27) and (28) of differential equations once with respect to q. Indeed, let us multiply eqn (28) by $\bar{g}'(q)/q$ and replace $\bar{g}(q)\bar{g}'(q)/q$ by $-\hat{f}(q)/4 \equiv -\hat{F}'(q)/4$ in the right-hand side, thanks to eqn (27); we get:

$$\bar{g}(q)\bar{g}'(q) - \frac{\bar{g}''(q)\bar{g}'(q)}{4} = -\frac{\hat{F}(q)\hat{F}'(q)}{4} \Rightarrow \bar{g}^2(q) - \frac{\bar{g}'^2(q)}{4} + \frac{\hat{F}^2(q)}{4} = Cst.$$

To determine the constant, take $q = 0$; then $\bar{g}(q) = 1/4$ and $\hat{F}(q) = 0$ as was seen above, and $\bar{g}'(q) = 0$ since the function $\bar{g}(q)$ is even. It follows that the value of the constant is $1/16$ so that:

$$4\bar{g}^2(q) - \bar{g}'^2(q) + \hat{F}^2(q) = \tfrac{1}{4} \Rightarrow \hat{F}(q) = \operatorname{sgn}(q)\sqrt{\tfrac{1}{4} + \bar{g}'^2(q) - 4\bar{g}^2(q)} \tag{29}$$

where sgn (q) denotes the sign of q. The choice of the sign before the radical here is dictated by the fact that for obvious physical reasons, $f(\eta)$ and its Fourier transform $\hat{f}(q)$ are "bell-shaped" functions so that the sign of the odd indefinite integral $\hat{F}(q)$ of $\hat{f}(q)$ is the same as that of q.

It is now trivial to eliminate the function $\hat{F}(q)$ between eqns (28) and (29_2); the result reads:

$$\bar{g}''(q) = 4\bar{g}(q)\left[1 - \frac{1}{|q|}\sqrt{\tfrac{1}{4} + \bar{g}'^2(q) - 4\bar{g}^2(q)}\right], \tag{30}$$

which is a second-order differential equation on the sole function $\bar{g}(q)$.

Unfortunately, integrating eqn (30) seems possible only by numerical means. When doing this, one only needs to take q in the range $[0, +\infty]$, since the function $\bar{g}(q)$ is even. There are then two possibilities: starting from some initial point $q_0 \ll 1$ (taking $q_0 = 0$ is impossible because of the singular character of the differential equation at that point) and integrating towards the right, or starting from $q_0 \gg 1$ and integrating towards the left. In the first case, one needs to know the behavior of $\bar{g}(q)$ for $q \to 0^+$ as an initial condition. The derivation of that behavior is somewhat involved and is presented in detail in Appendix B; the result is:

$$\bar{g}(q) = \frac{1}{4} + \frac{q^2}{4}\ln q + \lambda q^2 + \frac{q^4}{16}\ln^2 q + O(q^4|\ln q|) \quad \text{for} \quad q \to 0^+ \tag{31}$$

where λ is a constant, the value of which cannot be found by the present approach but in fact amounts to $\gamma/4 - (\ln 2)/2$ (where γ denotes Euler's constant), as is shown in Appendix A using Mouchrif's (1994) method. In the second case, it is the behavior of $\bar{g}(q)$ for $q \to +\infty$ that is required; the derivation is then much easier. Indeed, $\bar{g}(q)$ and $\bar{g}'(q)$ being assumed to vanish at infinity as above, eqn (30) simply reads $\bar{g}''(q) \sim 4\bar{g}(q)$ near infinity in a first approximation; excluding the divergent solution proportional to e^{2q}, one concludes that $\bar{g}(q)$ is proportional to e^{-2q}. A refined approximation may be found by reinserting that result into the radical of eqn (30) and rewriting that equation as:

$$\left(\frac{\bar{g}'(q)}{\bar{g}(q)}\right)' + \left(\frac{\bar{g}'(q)}{\bar{g}(q)}\right)^2 = 4 - \frac{2}{q} + O\left(\frac{e^{-4q}}{q}\right);$$

expanding $\bar{g}'(q)/\bar{g}(q)$ in powers of $1/q$, one easily finds then:

$$\frac{\bar{g}'(q)}{\bar{g}(q)} = -2 + \frac{1}{2q} - \frac{1}{16q^2} + O\left(\frac{1}{q^3}\right) \Rightarrow \bar{g}(q) = C\sqrt{q}\,e^{-2q}\left[1 + \frac{1}{16q} + O\left(\frac{1}{q^2}\right)\right] \quad \text{for} \quad q \to +\infty \tag{32}$$

where C is an unknown constant.

Once the function $\bar{g}(q)$ is known, $\hat{f}(q)$ is easily deduced from eqn (27). The function $\bar{f}(q)$ can also be obtained from the following formula, which is a consequence of eqns (11) and (29_2) and the properties $\bar{f}(0) = 1/4, \bar{f}'(0) = 0$ (see above):

$$\bar{f}(q) = \tfrac{1}{4} - \int_0^q \hat{F}(q')\,dq' = \tfrac{1}{4} - \left|\int_0^q \sqrt{\tfrac{1}{4} + \bar{g}'^2(q') - 4\bar{g}^2(q')}\,dq'\right|. \tag{33}$$

The functions $f(\eta)$ and $g(\eta)$ finally follow from Fourier inversion of $\hat{f}(q)$ and $\bar{g}(q)$:

$$f(\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(q)\,e^{-iq\eta}\,dq = \frac{1}{\pi}\int_0^{+\infty} \hat{f}(q)\cos(q\eta)\,dq; \tag{34}$$

$$g(\eta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{g}(q)\,e^{-iq\eta}\,dq = \frac{1}{\pi}\int_0^{+\infty} \bar{g}(q)\cos(q\eta)\,dq$$

where use has been made of the fact that $\hat{f}(q)$ and $\bar{g}(q)$ are even functions.

5.3. *Numerical procedure and results*

We decided in practice to start from some point $q_0 \gg 1$ and integrate towards the left rather than to start from $q_0 \ll 1$ and integrate towards the right. The motivation for that choice was based on the behavior of the solution of eqn (30) for $q \to +\infty$. Indeed, that solution then approximately becomes a linear combination of two components behaving (again approximately) like $e^{\pm 2q}$, and only one of these exponentials (e^{-2q}) is desired. Accurate numerical integration cannot be achieved if that desired exponential appears as the decreasing one, because slight errors in the initial data are then bound to generate an undesired increasing exponential which will completely conceal the true decreasing solution. This imposes the choice of an integration towards the left, for which the desired exponential e^{-2q} appears as the increasing one.

The value of q_0 chosen was 10. That of $\bar{g}(q_0)$ [or equivalently that of the constant C of eqn (32_2)] was adjusted in such a way as to get $\lim_{q\to 0^+} \bar{g}(q) \simeq \bar{g}(10^{-8}) = 1/4$ upon integration (using the Runge–Kutta method of order 4), the value of $\bar{g}'(q_0)$ being deduced from that of $\bar{g}(q_0)$ through eqn (32_1).

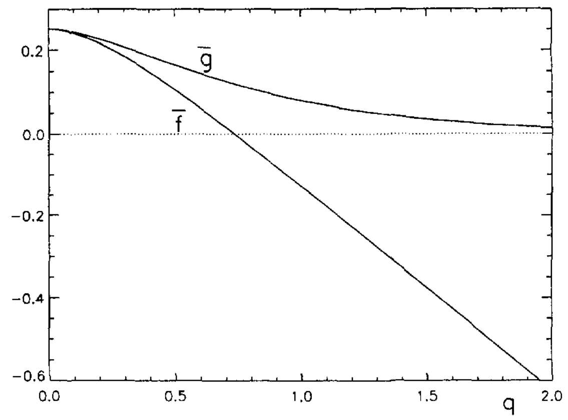

Fig. 7. The functions $\bar{f}(q)$ and $\bar{g}(q)$.

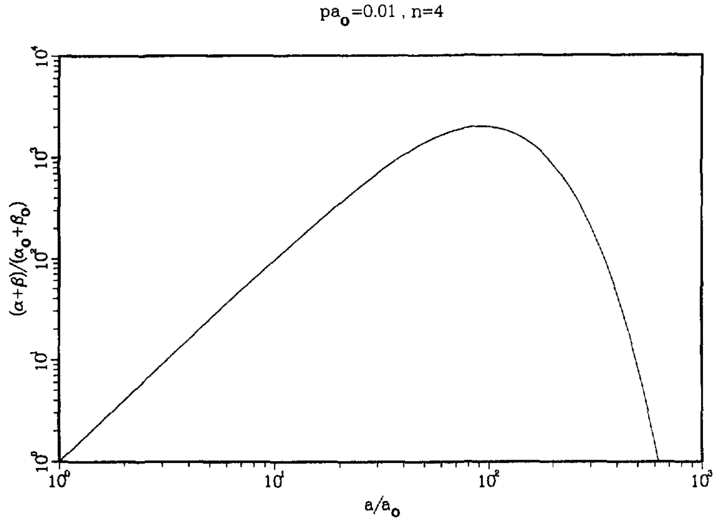

Fig. 8. The curve $(\alpha+\beta)/(\alpha_0+\beta_0) = f(a/a_0)$ in fatigue, for $pa_0 = 0.01$ and $n = 4$.

Figure 7 shows the functions $\bar{f}(q)$ and $\bar{g}(q)$ obtained numerically. From there, one immediately derives the values of q_s and the critical wavelength of bifurcation λ_s in brittle fracture:

$$q_s \simeq 0.925 \Rightarrow \lambda_s \simeq 6.793a. \tag{35}$$

The value of λ_s is remarkably close to that ($\simeq 5.2\ a$) found by Nguyen (1994) for the analogous (but mathematically much simpler) problem of an infinite tight membrane stuck onto a plate, except on an infinite strip subjected to an internal pressure. Also, Figs 8 and 9 show the curves $(\alpha\pm\beta)/(\alpha_0\pm\beta_0) = f(a/a_0)$ in fatigue for $pa_0 = 0.01$ and $n = 4$ (N.B. curves for other values of those parameters can be obtained through simple changes of origin and scale: see Section 4 above).

Finally, Fig. 10 shows the functions $f(\eta)$ and $g(\eta)$ in the range $0 \leqslant \eta \leqslant 10$. For larger values of η, one may safely use the following asymptotic formulae:

$$f(\eta) = \frac{1}{4\eta} + \frac{3}{2}\frac{\ln\eta}{\eta^3} + O\left(\frac{1}{\eta^3}\right);\quad g(\eta) = \frac{1}{4\eta^3} + \frac{3}{2}\frac{\ln\eta}{\eta^5} + O\left(\frac{1}{\eta^5}\right)\quad \text{for}\quad \eta \to +\infty. \tag{36}$$

The derivation of these expressions, which is based on eqn (31), is given in Appendix C.

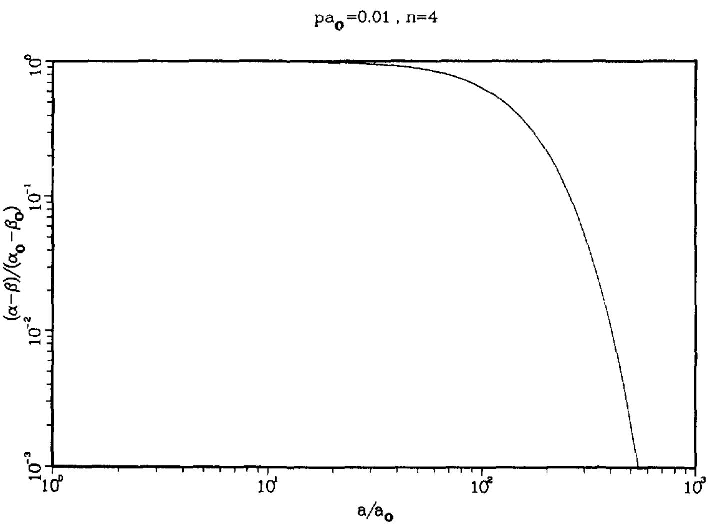

Fig. 9. The curve $(\alpha-\beta)/(\alpha_0-\beta_0) = f(a/a_0)$ in fatigue, for $pa_0 = 0.01$ and $n = 4$.

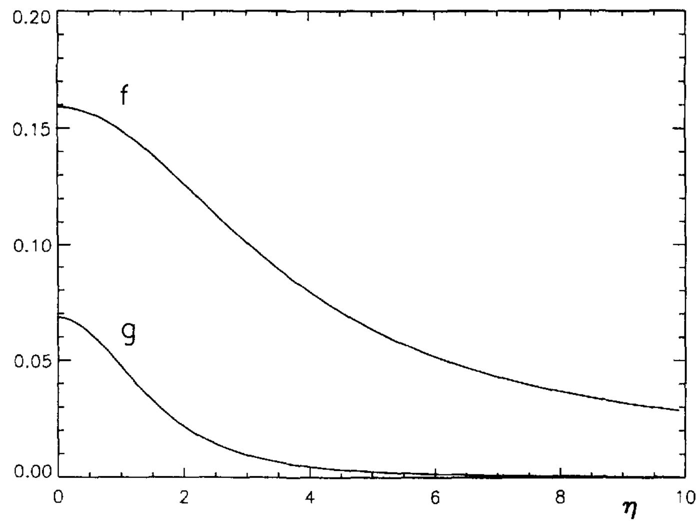

Fig. 10. The functions $f(\eta)$ and $g(\eta)$.

6. THE CRACK-FACE WEIGHT FUNCTION FOR A TENSILE TUNNEL-CRACK

We shall finally show that using the numerical values of the functions $\hat{F}(q)$ and $\bar{g}(q)$ obtained previously, one can now determine the crack-face weight function of a tunnel-crack loaded in pure mode I in an infinite body. It is reminded that this expression designates the function, which we shall denote $h(a; z^{\pm}, M)$, that gives the stress intensity factor at the point $z^{\pm}$ of the front $(+)$ or rear $(-)$ part of the crack front arising from a loading consisting of two opposite unit point forces perpendicular to the crack plane exerted on the points $M^{\pm}$ of the upper $(+)$ and lower $(-)$ crack lips.

There are of course a number of conventional techniques that could be used for that purpose. The finite element method is just one example. However, the method which will be employed here is more elegant in that the use it makes of numerical techniques is minimal; also, the results obtained are in all probability much more accurate.

In view of the obvious symmetries of the problem, one can assume the points of application $M^{\pm}$ of the point forces to be located at $(x, y = 0^{\pm}, z = 0)$ and the point of observation of the stress intensity factor at $(x = +a, y = 0, z)$ (Fig. 11). It is also appropriate to write the weight function in the two following forms:

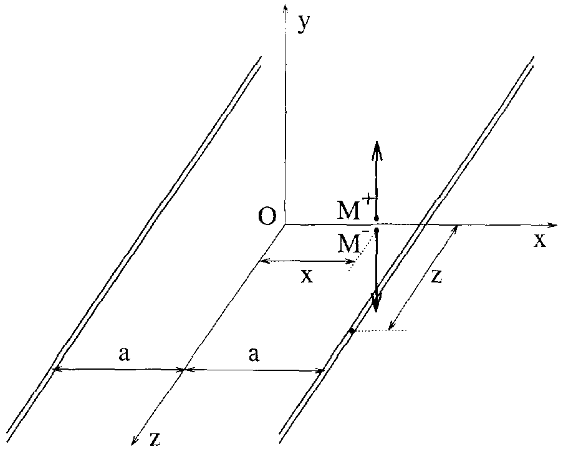

Fig. 11. Problem defining the crack-face weight function.

$$h(a;z^+,M) \equiv \frac{H(\xi,\eta)}{a^{3/2}} \equiv \frac{\sqrt{1-\xi^2}}{(1-\xi)^2+\eta^2}\frac{W(\xi,\eta)}{a^{3/2}}, \quad \xi \equiv \frac{x}{a}, \quad \eta \equiv \frac{z}{a}. \tag{37}$$

The introduction of $H(\xi,\eta)$ just serves to work with a dimensionless function. The introduction of $W(\xi,\eta)$, which was originally suggested by Rice (1989), is more subtle in the sense that the form of eqn (37_2) incorporates some known features of the singular behavior of the weight function, namely the fact that it tends to zero proportionally to $\sqrt{a \mp x}$ when M gets close to the front or rear part of the crack front ($x \to \pm a$), and also the fact that it behaves like $\sqrt{a-x}/((a-x)^2+z^2)$ for $(x,z) \to (a,0)$; in other words, $W(\xi,\eta)$ is more regular than the original weight function. This is illustrated by the following set of "boundary conditions", which show that $W(\xi,\eta)$ reduces to ordinary, non-singular functions on the lines $\xi = \pm 1$:

$$W(\xi = +1,\eta) = \frac{2}{\sqrt{\pi}}f(\eta); \quad W(\xi = -1,\eta) = \frac{2}{\sqrt{\pi}}(\eta^2+4)g(\eta). \tag{38}$$

These "boundary conditions" result from Rice's (1989) remark [combined with eqns (3)] that the limit of the quantity $h(a;z^+,M)/\sqrt{a \mp x}$ for $M \to (\pm a,0,0)$ is nothing else than $2\sqrt{(2/\pi)}Z(a;z^+,z' = 0^\pm)$. Numerical results will therefore be given for the function $W(\xi,\eta)$; however, the theoretical treatment will mainly concentrate on the function $H(\xi,\eta)$ because considering $W(\xi,\eta)$ would only lead to additional complexities and difficulties.

6.1. *Integro-differential equations on the function* H(ξ,η)

Let us consider the same first movement of the crack front as above (Fig. 6a). Then, for the loading which serves to define the function $h(a;z^+,M)$, the fundamental eqn (1) reads for an arbitrary point z^+ of the front part of the crack front, by eqn (3_2):

$$\delta k(z^+) = \int_{-\infty}^{+\infty} g\left(\frac{z-z'}{a}\right)h(a;z'^-,M)\frac{\varepsilon\,\mathrm{d}z'}{a^2}.$$

Now, the stress intensity factor $h(a;z'^-,M)$ at the point z'^- of the rear part of the crack front generated by the point forces at $M^\pm(x,y = 0^\pm,z = 0)$ is obviously equal to that generated at the point z'^+ of the front part of the front by point forces exerted on the points $M_1^\mp(-x,y = 0^\pm,z = 0)$, i.e. to $h(a;z'^+,M_1)$. This means that with the notation introduced in eqn (37_1):

$$\delta k(z^+) = \frac{\varepsilon}{a^{5/2}} \int_{-\infty}^{+\infty} g(\eta-\eta')H(-\xi,\eta')\,\mathrm{d}\eta'.$$

But $\delta k(z^+)$ is nothing else than the stress intensity factor corresponding to the situation where the crack width is $2a+\varepsilon$ while the loading remains unchanged, minus the original stress intensity factor, i.e. by eqn (37_1):

$$\varepsilon \frac{\partial}{\partial \varepsilon}\left[\frac{1}{(a+\varepsilon/2)^{3/2}} H\left(\frac{x+\varepsilon/2}{a+\varepsilon/2}, \frac{z}{a+\varepsilon/2}\right)\right]_{\varepsilon=0}$$
$$= \frac{\varepsilon}{a^{5/2}}\left\{-\tfrac{3}{4}H(\xi,\eta)+\frac{1}{2}\left[(1-\xi)\frac{\partial H}{\partial \xi}(\xi,\eta)-\eta\frac{\partial H}{\partial \eta}(\xi,\eta)\right]\right\}.$$

Comparing that expression with the preceding one, we get:

$$-\tfrac{3}{2}H(\xi,\eta)+(1-\xi)\frac{\partial H}{\partial \xi}(\xi,\eta)-\eta\frac{\partial H}{\partial \eta}(\xi,\eta) = 2\int_{-\infty}^{+\infty} g(\eta-\eta')H(-\xi,\eta')\,\mathrm{d}\eta'. \qquad (39)$$

Consider now the same second perturbation of the crack front as above (Fig. 6b), but assume here that the points z_1^+ and z_2^- have the same abscissa z along the Oz axis. Then eqn (1) reads at the point z^+:

$$\delta k(z^+) = -PV\int_{-\infty}^{+\infty}\frac{f(\eta-\eta')}{\eta-\eta'}H(\xi,\eta')\frac{\varepsilon\,\mathrm{d}\eta'}{a^{3/2}}+\int_{-\infty}^{+\infty}(\eta-\eta')g(\eta-\eta')H(-\xi,\eta')\frac{\varepsilon\,\mathrm{d}\eta'}{a^{3/2}}.$$

Now it is easy to see that if a' denotes the new half-width of the crack and $O'x'yz'$ the new "adapted" frame, one has:

$$\begin{cases} a' = a+O(\varepsilon^2) \\ x' = x+\varepsilon z+O(\varepsilon^2) \\ z' = z+\varepsilon(a-x)+O(\varepsilon^2), \end{cases}$$

and it follows that:

$$\delta k(z^+) \equiv \varepsilon\frac{\partial}{\partial \varepsilon}\left\{\frac{1}{a^{3/2}}H\left[\frac{x+\varepsilon z}{a},\frac{z+\varepsilon(a-x)}{a}\right]\right\}_{\varepsilon=0} = \frac{\varepsilon}{a^{3/2}}\left[\eta\frac{\partial H}{\partial \xi}(\xi,\eta)+(1-\xi)\frac{\partial H}{\partial \eta}(\xi,\eta)\right].$$

Comparison with the previous expression yields:

$$\eta\frac{\partial H}{\partial \xi}(\xi,\eta)+(1-\xi)\frac{\partial H}{\partial \eta}(\xi,\eta) = -PV\int_{-\infty}^{+\infty}\frac{f(\eta-\eta')}{\eta-\eta'}H(\xi,\eta')\,\mathrm{d}\eta'$$
$$+\int_{-\infty}^{+\infty}(\eta-\eta')g(\eta-\eta')H(-\xi,\eta')\,\mathrm{d}\eta'. \qquad (40)$$

6.2. *Partial differential and ordinary differential equations on the function* $\hat{\mathrm{H}}(\xi,\mathrm{q})$

Again, the presence of convolution products in eqns (39) and (40) is an invitation to take the Fourier transforms of these equations with respect to the variable η. This yields, as in Section 5,

$$(1-\xi)\frac{\partial \hat{H}}{\partial \xi}(\xi,q)+q\frac{\partial \hat{H}}{\partial q}(\xi,q)-\frac{\hat{H}(\xi,q)}{2}=2\bar{g}(q)\hat{H}(-\xi,q), \tag{41}$$

for eqn (39) and :

$$\frac{\partial^2 \hat{H}}{\partial \xi\,\partial q}(\xi,q)+(1-\xi)q\hat{H}(\xi,q)=\hat{F}(q)\hat{H}(\xi,q)+\bar{g}'(q)\hat{H}(-\xi,q) \tag{42}$$

for eqn (40).

Since we do not have a *single* partial differential equation at our disposal, but a *system* of such equations, it is possible to eliminate partial derivatives with respect to one variable in order to get an ordinary differential equation with respect to the other variable, which is of course much simpler to handle numerically. Thus, differentiation of eqn (41) with respect to ξ and elimination of the cross derivative $(\partial^2\hat{H}/\partial\xi\,\partial q)(\xi,q)$ with the aid of eqn (42), yields, discarding the arguments of the functions for simplicity,

$$(1-\xi)\frac{\partial^2 \hat{H}}{\partial \xi^2}-\frac{3}{2}\frac{\partial \hat{H}}{\partial \xi}+q[\hat{F}-(1-\xi)q]\hat{H}=2\bar{g}\frac{\partial \hat{H}^*}{\partial \xi}-q\bar{g}'\hat{H}^* \tag{43}$$

where

$$\hat{H}^*(\xi,q)\equiv\hat{H}(-\xi,q), \tag{44}$$

which is a differential equation with respect to ξ. Of course, in order to (numerically) solve that equation, one must supplement it with its counterpart at the point $(-\xi,q)$, which reads :

$$(1+\xi)\frac{\partial^2 \hat{H}^*}{\partial \xi^2}+\frac{3}{2}\frac{\partial \hat{H}^*}{\partial \xi}+q[\hat{F}-(1+\xi)q]\hat{H}^*=-2\bar{g}\frac{\partial \hat{H}}{\partial \xi}-q\bar{g}'\hat{H}. \tag{43'}$$

Also, initial conditions are needed at, or rather near (because of the singular character of the equations) the points $\xi=\pm 1$. These can be derived from eqns (38) ; indeed, near $\xi=-1$, one gets from eqn (38_2), with the aid of eqn (37_2) :

$$H(\xi,\eta)\equiv\sqrt{1-\xi^2}\,\frac{W(\xi,\eta)}{(1-\xi)^2+\eta^2}\sim\sqrt{2(1+\xi)}\,\frac{W(-1,\eta)}{\eta^2+4}=2\sqrt{\frac{2}{\pi}(1+\xi)}\;g(\eta),$$

so that

$$\hat{H}(\xi,q)\equiv\hat{H}^*(-\xi,q)\sim 2\sqrt{\frac{2}{\pi}(1+\xi)}\;\bar{g}(q)\quad\text{for}\quad \xi\to -1. \tag{45}$$

Near $\xi=+1$, one gets with the aid of eqn (38_1) :

$$H(\xi,\eta)\sim\sqrt{2(1-\xi)}\,\frac{W(+1,\eta)}{(1-\xi)^2+\eta^2}=2\sqrt{\frac{2}{\pi(1-\xi)}}\,\frac{1-\xi}{(1-\xi)^2+\eta^2}f(\eta).$$

Now $(1-\xi)/((1-\xi)^2+\eta^2)$ is a Lorentzian function of η with vanishingly small half-width $1-\xi$ and integral π ; it therefore tends to $\pi\delta(\eta)$ where δ denotes the Dirac distribution. It follows that :

$$H(\xi,\eta) \sim 2\sqrt{\frac{2\pi}{1-\xi}}\delta(\eta)f(0) = \sqrt{\frac{2}{\pi(1-\xi)}}\delta(\eta)$$

[by eqn (4_1)] and therefore that:

$$\hat{H}(\xi,q) \equiv \hat{H}^*(-\xi,q) \sim \sqrt{\frac{2}{\pi(1-\xi)}} \quad \text{for} \quad \xi \to +1. \tag{46}$$

One can also obtain a differential equation with respect to q by differentiating eqn (41) with respect to that variable and again eliminating the cross derivative with the aid of eqn (42); the result reads:

$$q\frac{\partial^2 \hat{H}}{\partial q^2} + \frac{1}{2}\frac{\partial \hat{H}}{\partial q} + (1-\xi)[\hat{F}-(1-\xi)q]\hat{H} = 2\bar{g}\frac{\partial \hat{H}^*}{\partial q} + (1+\xi)\bar{g}'\hat{H}^*, \tag{47}$$

whose counterpart at the point $(-\xi, q)$ is:

$$q\frac{\partial^2 \hat{H}^*}{\partial q^2} + \frac{1}{2}\frac{\partial \hat{H}^*}{\partial q} + (1+\xi)[\hat{F}-(1+\xi)q]\hat{H}^* = 2\bar{g}\frac{\partial \hat{H}}{\partial q} + (1-\xi)\bar{g}'\hat{H}. \tag{47'}$$

Since $H(\xi, \eta)$ and $\hat{H}(\xi, q)$ are obviously even functions of η and q, respectively, it suffices here to integrate over the interval $[0, +\infty[$. Anticipating that we shall start from some initial point $q_0 \gg 1$ and integrate towards the left, as we did for the function $\bar{g}(q)$, we see that we need to know the asymptotic behavior of $\hat{H}(\xi, q)$ and $\hat{H}^*(\xi, q)$ for $q \to +\infty$. It is easily found upon trial and error, using eqn (47), that $\hat{H}(\xi, q) \equiv \hat{H}^*(-\xi, q)$ is asymptotically proportional to $e^{-(1-\xi)q}$, and a more thorough analysis reveals that, more precisely,

$$\hat{H}(\xi,q) = \hat{H}_\infty(\xi)\,e^{-(1-\xi)q}\left\{1+O\left[\frac{e^{-2(1+\xi)q}}{\sqrt{q}}\right]\right\}.$$

In order to determine the function $\hat{H}_\infty(\xi)$, let us insert that asymptotic formula into the differential equation (43); using the fact that because of eqns (29_2) and (32), $\hat{F}(q) = \frac{1}{2} + O(e^{-4q})$, we get:

$$\{(1-\xi)[\hat{H}''_\infty(\xi)+2q\hat{H}'_\infty(\xi)+q^2\hat{H}_\infty(\xi)]-\tfrac{3}{2}[\hat{H}'_\infty(\xi)+q\hat{H}_\infty(\xi)]$$
$$+q[\tfrac{1}{2}-(1-\xi)q]\hat{H}_\infty(\xi)\}\,e^{-(1-\xi)q} = O[q^{3/2}\,e^{-(3+\xi)q}].$$

This yields the following two equations:

$$2(1-\xi)\hat{H}'_\infty(\xi)-\hat{H}_\infty(\xi) = 0\,;\,(1-\xi)\hat{H}''_\infty(\xi)-\tfrac{3}{2}\hat{H}'_\infty(\xi) = 0,$$

which admit a common solution proportional to $(1-\xi)^{-1/2}$; this means that:

$$\hat{H}(\xi,q) \equiv \hat{H}^*(-\xi,q) = \frac{\Gamma}{\sqrt{1-\xi}}e^{-(1-\xi)q}\left\{1+O\left[\frac{e^{-2(1+\xi)q}}{\sqrt{q}}\right]\right\} \quad \text{for} \quad q \to +\infty. \tag{48}$$

The constant Γ here is unknown, which means that we shall have to adjust it numerically in order to reach some "target" upon integration, as we did for the function $\bar{g}(q)$; again, that target will be be the values of the functions $\hat{H}(\xi, q)$ and $\hat{H}^*(\xi, q)$ at the point $q = 0$. To derive these values, one may take advantage of the well-known fact that the (uniform) stress intensity factor generated on the front part of the crack front by opposite unit line forces exerted on the lines $x = a\xi$, $y = 0^{\pm}$ (ξ being a parameter) is $(1/\sqrt{\pi a})\sqrt{(a+x)/(a-x)} = (1/\sqrt{\pi a})\sqrt{(1+\xi)/(1-\xi)}$; indeed this implies that:

$$\int_{-\infty}^{+\infty} H(\xi,\eta)\frac{a\,\mathrm{d}\eta}{a^{3/2}} = \frac{1}{\sqrt{\pi a}}\sqrt{\frac{1+\xi}{1-\xi}}$$

$$\Rightarrow \hat{H}(\xi, q=0) \equiv \hat{H}^*(-\xi, q=0) \equiv \int_{-\infty}^{+\infty} H(\xi,\eta)\,\mathrm{d}\eta = \frac{1}{\sqrt{\pi}}\sqrt{\frac{1+\xi}{1-\xi}}. \quad (49)$$

Let us finally note that the knowledge of the asymptotic behavior of $\hat{H}(\xi,q)$ and $\hat{H}^*(\xi,q)$ near the lines $\xi = \pm 1$ [eqns (45) and (46)] allows for an interesting check on the differential equation (47): that equation must automatically be satisfied to the dominant order in $1 \mp \xi$ when these functions are replaced by their asymptotic expressions. It is easy to check that this is indeed true: when one does so, one finds that eqn (47) reduces to eqn (28) for $\xi \to -1$ and to eqn (27) for $\xi \to +1$ (to the dominant order).

6.3. *Numerical procedure and results*

Although both groups of equations (43, 43′) and (47, 47′) can be used to compute $\hat{H}(\xi,q)$ and $\hat{H}^*(\xi,q)$, the method based on the latter group is preferable. Indeed integration of eqns (43, 43′) yields the values of these functions on the lines $q = Cst$; one must compute these values on all such lines before finally obtaining those of $H(\xi,\eta)$ [or more exactly $W(\xi,\eta)$] from Fourier inversion. In contrast, integration of eqns (47, 47′) yields the values of $\hat{H}(\xi,q)$ and $\hat{H}^*(\xi,q)$ on the lines $\xi = Cst$, and one only needs to know these values on one such line to get those of $H(\xi,\eta)$ [or $W(\xi,\eta)$] on that particular line by Fourier inversion; in other words, using eqns (47, 47′), one can determine the various functions $H(\xi,\eta) = f(\eta)$ ($\xi \equiv$ parameter) *independently of each other*, which results in an improved accuracy.

In practice, eqns (47, 47′) were integrated in the same way as eqn (30), i.e. towards the left, using $q_0 = 10$ as a starting point and stopping at $q = 10^{-8}$. The ratios $(\partial\hat{H}/\partial q)(\xi,q_0)/\hat{H}(\xi,q_0)$ and $(\partial\hat{H}^*/\partial q)(\xi,q_0)/\hat{H}^*(\xi,q_0)$ were taken equal to $-1+\xi$ and $-1-\xi$ respectively, as imposed by eqn (48).† The adjustment of the constant Γ was a trivial matter, in constrast to that of the constant C involved in the calculation of the function $\bar{g}(q)$ (see Section 5.3); indeed, since eqns (47, 47′) are linear [unlike eqn (30)], it was sufficient to first choose Γ arbitrarily and integrate them to get the corresponding value of $\hat{H}(\xi, q=0) \simeq \hat{H}(\xi, q=10^{-8})$, and then multiply Γ and all values of $\hat{H}(\xi,q)$ by the "scaling" factor ensuring that eqn (49) be satisfied.

Once $\hat{H}(\xi,q)$ was known, two possibilities could be envisaged for the final calculation of the desired function $W(\xi,\eta)$: either Fourier-invert $\hat{H}(\xi,q)$ to get $H(\xi,\eta)$ and then get $W(\xi,\eta)$ from eqn (37_2), or evaluate $\hat{W}(\xi,q)$ from the formula:

$$\hat{W}(\xi,q) = -\frac{1}{\sqrt{1-\xi^2}}\left[(1-\xi)^2\hat{H}(\xi,q) - \frac{\partial^2\hat{H}}{\partial q^2}(\xi,q)\right], \quad (50)$$

which is a straightforward consequence of eqn (37_2), and then get $W(\xi,\eta)$ by Fourier inversion. The second solution was preferred because Fourier inversion of $\hat{H}(\xi,q)$ would have required values of that function well beyond $q_0 = 10$ to be known, at least for the largest value of ξ envisaged (0.8), since its decrease, which is proportional to $e^{-(1-\xi)q}$ according to eqn (48), is rather slow then. [That problem does not arise for the function $\hat{W}(\xi,q)$ since the latter is easily verified, using eqns (48) and (50), to decrease proportionally to $e^{-(3+\xi)q}/\sqrt{q}$, i.e. much more quickly, for $q \to +\infty$.]

† The term $O(e^{-2(1-\xi)q}/\sqrt{q})$ which appears in that equation was discarded because for the values of ξ considered, which were all between -0.8 and $+0.8$, it was small for $q = q_0 = 10$.

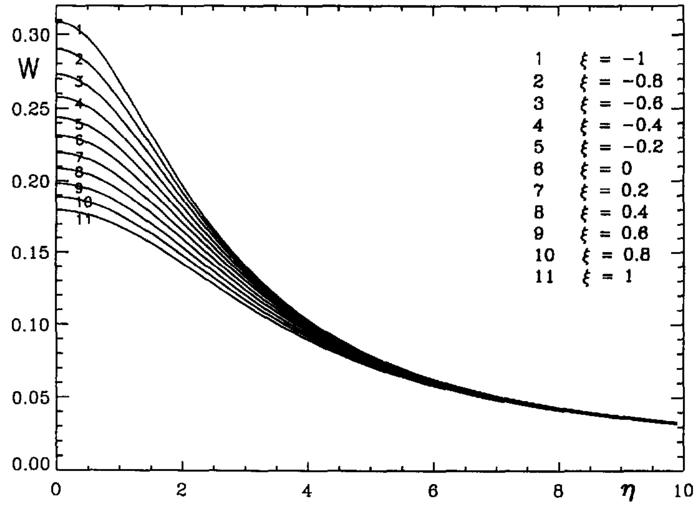

Fig. 12. The curves $W(\xi, \eta) = f(\eta)$ for various values of ξ.

Figure 12 shows the curves $W(\xi, \eta) = f(\eta)$ obtained in that way for several values of ξ. These curves strongly suggest that the asymptotic behavior of that function for $\eta \to +\infty$ might be the same whatever the value of ξ. An extra argument in favor of that conjecture is that according to eqns (38) and (36), $W(\xi = -1, \eta) \sim W(\xi = +1, \eta) \sim 1/(2\sqrt{\pi\eta})$ for $\eta \to +\infty$. In fact, it is not difficult to check that whatever the value of ξ,

$$W(\xi, \eta) \sim \frac{1}{2\sqrt{\pi\eta}} \quad \text{for}\, \eta \to +\infty \tag{51}$$

(which comes as a nice complement to the numerical results presented in Fig. 12). Indeed, following the same type of reasoning as in Appendix C, one sees that eqn (51) is equivalent to $\hat{W}(\xi, q) \sim -(\ln q)/\sqrt{\pi}$ for $q \to 0^+$ or, because of eqns (49) and (50), to:

$$\hat{H}(\xi, q) \equiv \hat{H}(-\xi, q) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{1+\xi}{1-\xi}} + \sqrt{\frac{1-\xi^2}{\pi}}\frac{q^2}{2}\ln q \cdot [1+\varepsilon(q)] \quad \text{for}\, q \to 0^+ \tag{52}$$

where $\lim_{q \to 0^+} \varepsilon(q) = 0$. Now one readily verifies that this asymptotic formula is correct by inserting it into the differential equations (43) and (47), and checking that they are then identically satisfied to the dominant order in q.

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APPENDIX A: MOUCHRIF'S METHOD FOR EVALUATING THE FUNCTION $\bar{g}(q)$—CALCULATION OF THE CONSTANT λ OF EQN (31)

In Mouchrif's (1994) thesis, the method used to calculate the function $\bar{g}(q)$ [which is connected to the limit of the Fourier transform $\hat{H}(\xi,q)$ of the weight function for $\xi \to -1$, as shown by eqn (45)] was to numerically solve the following integral equation on the function $\hat{H}(\xi,q)$, which was deduced from the work of Bui (1978):

$$PV\int_{-1}^{+1}\left(\{\operatorname{sgn}(\xi-\xi')K_0'(q|\xi'-\xi|)+K_0'[q(1-\xi)]\}\frac{\partial \hat{H}}{\partial \xi}(\xi',q)-qK_0(q|\xi'-\xi|)\hat{H}(\xi',q)\right)\mathrm{d}\xi'=0 \quad (\forall \xi \in]-1,+1[) \tag{A1}$$

where K_0 denotes the classical Bessel function of order 0. (In fact, that procedure does not only yield the limiting values of the Fourier transform of the weight function for $\xi \to -1$, but *all* values of that Fourier transform.) As mentioned in the text, the resulting numerical values are very close to those obtained by the method described in Section 5.

One of the referees suggested that $\hat{H}(\xi,q)$ could be obtained by solving the following equation, which he derived from Bueckner's (1987) work:

$$\int_{-1}^{+1}K_0(q|\xi'-\xi|)\hat{H}(\xi',q)\,\mathrm{d}\xi'=C_1(q)\,\mathrm{e}^{q\xi}+C_2(q)\,\mathrm{e}^{-q\xi} \tag{A2}$$

where $C_1(q)$ and $C_2(q)$ are unknown functions. In fact, although Mouchrif did not notice that his equation could be put in such a nice and simple form, eqns (A1) and (A2) are equivalent. Indeed eqn (A2) is equivalent to:

$$\frac{\partial^2\Phi}{\partial \xi^2}(\xi,q)-q^2\Phi(\xi,q)=0,\quad \Phi(\xi,q)\equiv\int_{-1}^{+1}K_0(q|\xi'-\xi|)\hat{H}(\xi',q)\,\mathrm{d}\xi'; \tag{A2'}$$

evaluating then $\partial^2\Phi/\partial\xi^2$ [paying attention to the fact that $K_0(x)$ diverges logarithmically for $x \to 0^+$], using the relation $K_0''(x)+K_0'(x)/x-K_0(x)=0$ and integrating by parts, one readily gets eqn (A1).

It was also suggested by the same referee that eqn (A2) could be used to derive the asymptotic behavior of $\hat{H}(\xi,q)$ for $q \to 0^+$, and in particular the value of the unknown constant λ appearing in eqn (31). The procedure is as follows. First, using the asymptotic expression of $K_0(x)$ for $x \to 0^+$, namely

$$K_0(x)=-\left(1+\frac{x^2}{4}\right)\ln\frac{x}{2}-\gamma+(1-\gamma)\frac{x^2}{4}+O(x^4|\ln x|)$$

where γ denotes Euler's constant [see, e.g. Gradshteyn and Ryzhik (1965)], and assuming $\hat{H}(\xi,q)$ to be bounded for $q \to 0^+$, one gets the following asymptotic expression of the function $\Phi(\xi,q)$ defined by eqn (A$'2_2$) for small values of q:

$$\begin{aligned}\Phi(\xi,q)=&-\left(\ln\frac{q}{2}+\gamma\right)\int_{-1}^{+1}\hat{H}(\xi',q)\,\mathrm{d}\xi'-\int_{-1}^{+1}\ln(|\xi'-\xi|)\hat{H}(\xi',q)\,\mathrm{d}\xi'\\&-\frac{q^2}{4}\left(\ln\frac{q}{2}+\gamma-1\right)\int_{-1}^{+1}(\xi'-\xi)^2\hat{H}(\xi',q)\,\mathrm{d}\xi'\\&-\frac{q^2}{4}\int_{-1}^{+1}(\xi'-\xi)^2\ln(|\xi'-\xi|)\hat{H}(\xi',q)\,\mathrm{d}\xi'+O(q^4|\ln q|).\end{aligned}$$

Evaluating then the quantity $\partial^2\Phi(\xi,q)/\partial\xi^2-q^2\Phi(\xi,q)$, one finds that eqn (A$'2_1$) reads

$$\frac{\partial}{\partial \xi}\left[PV\int_{-1}^{+1}\frac{\hat{H}(\xi',q)}{\xi'-\xi}\,d\xi'\right] = -\frac{q^2}{2}\left(\ln\frac{q}{2}+\gamma-\frac{1}{2}\right)\int_{-1}^{+1}\hat{H}(\xi',q)\,d\xi' - \frac{q^2}{2}\int_{-1}^{+1}\ln(|\xi'-\xi|)\hat{H}(\xi',q)\,d\xi' + O(q^4|\ln q|) \tag{A3}$$

for $q \to 0^+$.

Now we know, by eqn (52) of the text, that

$$\hat{H}(\xi,q) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{1+\xi}{1-\xi}} + O(q^2|\ln q|). \tag{A4}$$

It follows that if we replace $\hat{H}(\xi',q)$ by $(1/\sqrt{\pi})\sqrt{(1+\xi')/(1-\xi')}$ in the right-hand side of eqn (A3), the error made is $O(q^4\ln^2 q)$. Making that substitution and calculating the resulting integrals, one gets:

$$\frac{\partial}{\partial \xi}\left[PV\int_{-1}^{+1}\frac{\hat{H}(\xi',q)}{\xi'-\xi}\,d\xi'\right] = \frac{\sqrt{\pi}q^2}{2}(\xi-\ln q-\gamma+\tfrac{1}{2}+2\ln 2)+O(q^4\ln^2 q). \tag{A5}$$

The last step consists in looking for a series expansion of $\hat{H}(\xi,q)$ of the form [suggested by eqn (52)]:

$$\hat{H}(\xi,q) = \frac{1}{\sqrt{\pi(1-\xi^2)}}\sum_{n=0}^{+\infty} a_n(q)\xi^n. \tag{A6}$$

Substitution of that expansion in the integral appearing in eqn (A5) requires the calculation of the integrals:

$$I_n \equiv PV\int_{-1}^{+1}\frac{\xi'^n\,d\xi'}{(\xi'-\xi)\sqrt{1-\xi'^2}} \quad (n = 0,1,2,\ldots).$$

This is feasible for all values of n, and it is found that I_n is a polynomial expression of ξ of degree $n-1$. This implies that the term $a_n(q)\xi'^n$ in the expression of $\hat{H}(\xi',q)$ generates, in the left-hand side of eqn (A5), a term of degree $n-2$ with respect to ξ; comparison with the right-hand side of the same equation then reveals that for $n \geqslant 4 (\Rightarrow n-2 \geqslant 2)$, the coefficient $a_n(q)$ is necessarily $O(q^4\ln^2 q)$. It follows that if terms of that order are to be disregarded, it is sufficient to consider only the first four terms of the series (A6), corresponding to $n = 0$, 1, 2 and 3. Equation (A5) then becomes, upon calculation of the integrals I_0, I_1, I_2, I_3:

$$a_2(q)+2a_3(q)\xi = \frac{q^2}{2}(\xi-\ln q-\gamma+\tfrac{1}{2}+2\ln 2)+O(q^4\ln^2 q) \Rightarrow \begin{cases} a_2(q) = \dfrac{q^2}{2}(-\ln q-\gamma+\tfrac{1}{2}+2\ln 2)+O(q^4\ln^2 q) \\[2ex] a_3(q) = \dfrac{q^2}{4}+O(q^4\ln^2 q) \end{cases}. \tag{A7}$$

Furthermore, eqns (45) and (46) of the text imply that:

$$\begin{cases} \lim\limits_{\xi\to+1}\sqrt{\dfrac{\pi(1-\xi)}{2}}\,\hat{H}(\xi,q) = 1 \\[2ex] \lim\limits_{\xi\to-1}\dfrac{1}{2}\sqrt{\dfrac{\pi}{2(1+\xi)}}\,\hat{H}(\xi,q) = \bar{g}(q) \end{cases}.$$

It is easy to see that these equations bear the following consequences on the coefficients $a_n(q)$:

$$\begin{cases} a_0(q)+a_1(q)+a_2(q)+a_3(q) = 2+O(q^4\ln^2 q) \\ a_0(q)-a_1(q)+a_2(q)-a_3(q) = O(q^4\ln^2 q) \\ a_1(q)-2a_2(q)+3a_3(q) = 4\bar{g}(q)+O(q^4\ln^2 q) \end{cases} \Rightarrow \begin{cases} a_0(q) = 1-a_2(q)+O(q^4\ln^2 q) \\ a_1(q) = 1-a_3(q)+O(q^4\ln^2 q) \\ \bar{g}(q) = \dfrac{1}{4}+\dfrac{q^2}{4}\ln q+\left(\dfrac{\gamma}{4}-\dfrac{\ln 2}{2}\right)q^2+O(q^4\ln^2 q) \end{cases}. \tag{A8}$$

Equation (A8$_3$) is compatible with eqn (31) of the text, with $\lambda = \gamma/4-(\ln 2)/2$. Also, eqns (A7) and (A8$_{1,2}$) imply that the term proportional to $q^2\ln q$ in the expression of $\hat{H}(\xi,q)$ is $\sqrt{(1-\xi^2)/\pi}(q^2/2)\ln q$, in agreement with eqn (52) of the text.

It is clear that one could derive the expression of the function $\hat{H}(\xi,q)$ for $q \to 0^+$ up to any desired degree of accuracy in a similar way.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF THE FUNCTION $\bar{g}(q)$ FOR $q \to 0^+$

The aim of this Appendix is to derive eqn (31) of the text. The proof will not use the second-order differential equation (30), but the following more convenient (for the present purpose) third-order equation:

$$\bar{g}'''(q)+\frac{\bar{g}''(q)}{q}-\frac{\bar{g}'(q)\bar{g}''(q)}{\bar{g}(q)}-16\frac{\bar{g}^2(q)\bar{g}'(q)}{q^2}-4\frac{\bar{g}(q)}{q}=0, \tag{B1}$$

which can be obtained either by differentiating eqn (30) or by directly eliminating $\hat{F}(q)$ between eqns (27) and (28).

The first step consists in distinguishing between "major" and "minor" terms in that equation; the procedure for obtaining that decomposition is essentially heuristic, since the latter depends upon the asymptotic behavior of $\bar{g}(q)$ for $q \to 0^+$, which is precisely what is looked for. Upon trial and error, we are finally led to propose the following equivalent form of eqn (B1):

$$\bar{g}'''(q)+\frac{\bar{g}''(q)}{q}-\frac{\bar{g}'(q)}{q^2}-\frac{1}{q}=\varphi(q)\equiv\frac{\bar{g}'(q)\bar{g}''(q)}{\bar{g}(q)}+\frac{16\bar{g}^2(q)-1}{q^2}\bar{g}'(q)+\frac{4\bar{g}(q)-1}{q} \tag{B2}$$

where all "major" terms are gathered in the left-hand side and all "minor" ones in the right-hand side.

We now integrate eqn ($B2_1$) three times for $q>0$, formally considering the function $\varphi(q)$ as known and using the fact that $\bar{g}'''(q)+\bar{g}''(q)/q-\bar{g}'(q)/q^2-1/q=\{(1/q)[q\bar{g}'(q)]'-\ln q\}'$:

$$\begin{aligned}
&\frac{1}{v}[v\bar{g}'(v)]'-\ln v=\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w+\alpha\\
&\quad\Rightarrow [v\bar{g}'(v)]'=v\ln v+v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w+\alpha v\\
&\quad\Rightarrow u\bar{g}'(u)=\frac{u^2}{2}(\ln u-\tfrac{1}{2})+\int_{u_0}^{u}v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w+\frac{\alpha u^2}{2}+\beta\\
&\quad\Rightarrow \bar{g}'(u)=\frac{u}{2}(\ln u-\tfrac{1}{2})+\frac{1}{u}\int_{u_0}^{u}v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w+\frac{\alpha u}{2}+\frac{\beta}{u}\\
&\quad\Rightarrow \bar{g}(q)=\frac{q^2}{4}(\ln q-1)+\int_{q_0}^{q}\frac{\mathrm{d}u}{u}\int_{u_0}^{u}v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w+\frac{\alpha q^2}{4}+\beta\ln q+\gamma\\
&\quad\equiv\frac{q^2}{4}\ln q+\lambda q^2+\beta\ln q+\gamma+\int_{q_0}^{q}\frac{\mathrm{d}u}{u}\int_{u_0}^{u}v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w
\end{aligned} \tag{B3}$$

where α, β, γ, $\lambda\equiv(\alpha-1)/4$, q_0, u_0, v_0 are constants; in fact, in spite of appearances, $\bar{g}(q)$ depends only on three arbitrary constants here, because changing the lower bounds of integration v_0, u_0, q_0 is equivalent to changing the integration constants α, β, γ.

The asymptotic behavior of $\bar{g}(q)$ for $q\to 0^+$ will now be deduced from eqn (B3) through successive "iterations". The first one will consist in introducing a set of minimal assumptions on the behavior of that function, examining their consequences on that of the function $\varphi(q)$ defined by eqn ($B2_2$), and finally deducing from there and eqn (B3) a more accurate description of the behavior of $\bar{g}(q)$. In a second iteration, we shall reinsert that refined information into the expression of $\varphi(q)$ in order to refine those concerning that function, and then re-use eqn (B3) to further refine those about $\bar{g}(q)$, and so on. The basic reason why the whole procedure works is that the triple integration which appears in eqn (B3) is a "regularizing" process, which means that the output is a less singular function than the input.

Iteration 1

Our set of initial hypotheses is:

$$\bar{g}(0)=\tfrac{1}{4};\bar{g}'(0)=0;|\bar{g}''(q)|\leqslant\frac{C}{q}\quad\text{for}\quad q\to 0^- \tag{B4}$$

where C is a positive constant; the third assumption here is in fact somewhat overpessimistic, since if $\bar{g}''(q)$ behaved as $1/q$, $\bar{g}'(q)$ would not exist at the point $q=0$. It is easy to see that eqn (B4) implies that $\varphi(q)$ is also bounded by C_1/q where C_1 is another positive constant. Taking then $v_0>0$ in order to avoid a possible divergence of the integral $\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w$, we see that the latter is bounded by $C_2|\ln v|$. Since the integral of the logarithm is convergent at the point 0, we can now take $u_0=0$ and see that $\int_0^u v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w$ is bounded by $C_3u^2|\ln u|$, and then again take $q_0=0$ and conclude that $\int_0^q \mathrm{d}u/u\int_0^u v\,\mathrm{d}v\int_{v_0}^{v}\varphi(w)\,\mathrm{d}w$ is bounded by $C_4q^2|\ln q|$. Inserting that result into eqn (B3), we get:

$$\bar{g}(q)=\beta\ln q+\gamma+O(q^2|\ln q|).$$

But eqn ($B4_1$) then implies that $\beta=0$ and $\gamma=1/4$. Thus eqn (B3) takes the form

$$\bar{g}(q) = \frac{1}{4} + \frac{q^2}{4}\ln q + \lambda q^2 + \int_0^q \frac{\mathrm{d}u}{u}\int_0^u v\,\mathrm{d}v \int_{v_0}^{v} \varphi(w)\,\mathrm{d}w. \tag{B3'}$$

Since we now know that the triple integral here is bounded by $C_4 q^2|\ln q|$, we may conclude the iteration by stating that:

$$\bar{g}(q) = \tfrac{1}{4} + O(q^2|\ln q|);\quad \bar{g}'(q) = O(q|\ln q|);\quad \bar{g}''(q) = O(|\ln q|) \quad \text{for} \quad q \to 0^+. \tag{B5}$$

Iteration 2

Inserting eqns (B5) into the expression (B2$_2$) of $\varphi(q)$, one sees that it is bounded by $C_5 q \ln^2 q$. Thus the integral of that function is convergent at the point 0, so that we can take $v_0 = 0$ hereafter. One then easily sees that $\int_0^q \mathrm{d}u/u \int_0^u v\,\mathrm{d}v \int_0^v \varphi(w)\,\mathrm{d}w$ is bounded by $C_6 q^4 \ln^2 q$. Using eqn (B3′), we conclude the iteration by stating that:

$$\bar{g}(q) = \tfrac{1}{4} + \frac{q^2}{4}\ln q + \lambda q^2 + \int_0^q \frac{\mathrm{d}u}{u}\int_0^u v\,\mathrm{d}v \int_0^v \varphi(w)\,\mathrm{d}w \tag{B3''}$$

[which is just eqn (B3′) with $v_0 = 0$] and that:

$$\bar{g}(q) = \tfrac{1}{4} + \frac{q^2}{4}\ln q + \lambda q^2 + O(q^4\ln^2 q);\quad \bar{g}'(q) = \frac{q}{2}\ln q + (\tfrac{1}{4} + 2\lambda)q + O(q^3\ln^2 q);$$

$$\bar{g}''(q) = \frac{\ln q}{2} + \tfrac{3}{4} + 2\lambda + O(q^2\ln^2 q) \quad \text{for} \quad q \to 0^+. \tag{B6}$$

Iteration 3

Using eqns (B6) and retaining only the dominant terms, one sees that $\varphi(q) = 2q\ln^2 q + O(q|\ln q|)$; inserting that result into eqn (B3″), one finally gets eqn (31) of the text.

APPENDIX C: ASYMPTOTIC BEHAVIOR OF THE FUNCTIONS $f(\eta)$ AND $g(\eta)$ FOR $\eta \to +\infty$

The asymptotic behavior of $f(\eta)$ and $g(\eta)$ for $\eta \to +\infty$ can be deduced from that of $\bar{g}(q)$ for $q \to 0^+$. Indeed, with regard to $g(\eta)$, repeated integration by parts† of the integral in eqn (34$_2$) yields:

$$\begin{aligned}
g(\eta) &= \frac{1}{\pi}\left[\bar{g}(q)\frac{\sin(q\eta)}{\eta}\right]_{q=0}^{q=+\infty} - \frac{1}{\pi\eta}\int_0^{+\infty} \bar{g}'(q)\sin(q\eta)\,\mathrm{d}q \\
&= \frac{1}{\pi\eta}\left[\bar{g}'(q)\frac{\cos(q\eta)}{\eta}\right]_{q=0}^{q=+\infty} - \frac{1}{\pi\eta^2}\int_0^{+\infty} \bar{g}''(q)\cos(q\eta)\,\mathrm{d}q \\
&= -\frac{1}{\pi\eta^2}\left[\bar{g}''(q)\frac{\sin(q\eta)}{\eta}\right]_{q=0}^{q=+\infty} + \frac{1}{\pi\eta^3}\int_0^{+\infty} \bar{g}'''(q)\sin(q\eta)\,\mathrm{d}q \\
&= \frac{1}{\pi\eta^3}\int_0^{+\infty} \bar{g}'''\left(\frac{q'}{\eta}\right)\sin q'\,\frac{\mathrm{d}q'}{\eta} \qquad (q\eta \equiv q');
\end{aligned}$$

the bracketed terms here vanish because of various properties of $\bar{g}(q)$ mentioned above. Now eqn (31) implies that:

$$\bar{g}'''(q) = \frac{1}{2q} + \tilde{g}(q), \quad \text{where} \quad \tilde{g}(q) = \frac{3q}{2}\ln^2 q + O(q|\ln q|) \quad \text{for} \quad q \to 0^+;$$

insertion of that result into the preceding expression and further integration by parts then yields

$$\begin{aligned}
g(\eta) &= \frac{1}{2\pi\eta^3}\int_0^{+\infty} \frac{\sin q'}{q'}\,\mathrm{d}q' + \frac{1}{\pi\eta^4}\int_0^{+\infty} \tilde{g}\left(\frac{q'}{\eta}\right)\sin q'\,\mathrm{d}q' \\
&= \frac{1}{4\eta^3} - \frac{1}{\pi\eta^4}\left[\tilde{g}\left(\frac{q'}{\eta}\right)\cos q'\right]_{q'=0}^{q'=+\infty} + \frac{1}{\pi\eta^4}\int_0^{+\infty} \frac{1}{\eta}\tilde{g}'\left(\frac{q'}{\eta}\right)\cos q'\,\mathrm{d}q' \\
&= \frac{1}{4\eta^3} + \frac{1}{\pi\eta^5}\left[\tilde{g}'\left(\frac{q'}{\eta}\right)\sin q'\right]_{q'=0}^{q'=+\infty} - \frac{1}{\pi\eta^5}\int_0^{+\infty} \frac{1}{\eta}\tilde{g}''\left(\frac{q'}{\eta}\right)\sin q'\,\mathrm{d}q' \\
&= \frac{1}{4\eta^3} - \frac{1}{\pi\eta^5}\int_0^{+\infty} \frac{1}{\eta}\tilde{g}''\left(\frac{q'}{\eta}\right)\sin q'\,\mathrm{d}q'.
\end{aligned}$$

† Note that we implicitly use here the fact that the only singularity of the function $\bar{g}(q)$ is at the point 0 [because the differential equation (30) is singular only there].

Since $q'/\eta \to 0$ for $\eta \to +\infty$, whatever the value of q', one may replace $\tilde{g}''(q'/\eta)$ here by its asymptotic expression, namely $3\ln(q'/\eta)/(q'/\eta) + O(\eta/q') = -3(\eta \ln \eta)/q' + O(\eta|\ln q'|/q')$, and the preceding equation becomes

$$g(\eta) = \frac{1}{4\eta^3} + \frac{3\ln\eta}{\pi\eta^5}\int_0^{+\infty}\frac{\sin q'}{q'}\,dq' + O\left(\frac{1}{\eta^5}\right) = \frac{1}{4\eta^3} + \frac{3}{2}\frac{\ln\eta}{\eta^5} + O\left(\frac{1}{\eta^5}\right),$$

which is eqn (36_2) of the text.

To derive the asymptotic behavior of the function $f(\eta)$ (for $\eta \to +\infty$), one must first obtain that of $\hat{f}(q)$ (for $q \to 0^+$), which readily follows from eqns (27) and (31):

$$\hat{f}(q) = -\frac{\ln q}{2} - \frac{1}{4} - 2\lambda - \frac{3q^2}{4}\ln^2 q + O(q^2|\ln q|) \quad \text{for} \quad q \to 0^+.$$

Equation (36_1) of the text then follows from a reasoning quite analogous to that just presented for the function $g(\eta)$.